APPLICATIONS OF THE STOCHASTIC CONTROL APPROACH TO THE EGYPTIAN ECONOMY

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Abstract

This study outlines the “Stochastic Control Approach” technique and presents examples that illustrate the use of this method on two aspects of the Egyptian economy. Specifically, two empirical applications using data on Egypt are discussed in detail: optimal allocation of investment (Part I); and optimal level of reserves (Part II). In both applications, the estimated optimal levels are compared to the actual ones with useful insights for investment distribution for maximizing GDP growth, as well as, hedging sudden stops. In the process, we delineate some of the applications and highlight areas where further applications of the approach may be specifically fruitful.

ملخص

تضع هذه الدراسة معالم أسلوب "نهج التحكم التصادفي" وتقدم أمثلة توضح استخدام هذه الطريقة في جانبين من جوانب الاقتصاد المصري. على وجه التحديد، تناقش الدراسة بالتفصيل تطبيقين تجريبيين باستخدام بيانات عن مصر، وهما: التخصيص الأمثل للاستثمار (في الجزء الأول من الدراسة)، والمستوى الأمثل للاحتياطيات (في الجزء الثاني). وفي كلا الجزأين، تُقارَن المستويات المثلى المقدرة بالمستويات الفعلية مع تقديم مقترحات مفيدة لتوزيع الاستثمار لتعظيم نمو الناتج المحلي الإجمالي، وكذلك للتحوط من التوقفات الاقتصادية المفاجئة. وفي معرض ذلك، نحدد بعض التطبيقات ونلقي الضوء على المجالات التي قد تكون فيها التطبيقات الإضافية للنهج مفيدة بصفة خاصة.

JEL Classifications: G01, O2

Keywords: Investments, GDP, Stochastic Optimal Control, Growth Theory, Reserves, Consumption, Stochastic Control, the Martingale Optimality Principle.

PART I: AN APPLICATION OF THE STOCHASTIC CONTROL APPROACH TO: OPTIMAL ALLOCATIONS OF INVESTMENTS FOR MAXIMUM GDP GROWTH

1. INTRODUCTION

Investment is undoubtedly central for growth and job creation and more importantly, allocation of investments does matter for policymaking and social welfare. However, when there exist limited resources, questions about optimal allocation arise. Investment strategy can target a wide range of investment projects in different sectors, which all could likely be beneficial—with varying degrees—for growth and employment. However, due to limited resources, trade-offs arise so that there has to be strategic decisions to which sectoral investments should be allocated. Yet, optimal decision making in the context of resource allocation among different sectors requires knowledge of the returns on each of the possible investment distribution. In this spirit, this study primarily aims to guide future investment allocation by explaining the weaknesses and the pitfalls of past trends. This study uses a different approach to give us some insights into why the current investment allocation is not likely to be optimal. In contrast to the conventional methods of studying the relationship between investments and economic growth, this study employs the methods of stochastic control developed by Merton (1971, 1992) and Abutaleb and Papaionnou (2019) to evaluate the actual distribution of investments to that of its optimal one.

The study contributes to the ongoing debate on promoting investments in productive sectors to maximize benefits to growth and employment. We attempt to evaluate the current allocation of investments to highlight some room for improvements in situations when the returns on investment for certain sectors are low. Specifically, the objective of this study is to investigate the relationship between investments and GDP growth in Egypt and to devise a mechanism for the maximization of GDP growth. The analysis is done for five sectors: (1) agriculture, (2) manufacturing, (3) construction, (4) communications, and (5) tourism. Hence, the main objective of this study is to evaluate the historical distribution of investments by answering three key questions: i) what is the optimal distribution of investments of the five sectors under study? ii) Within the manufacturing sector, what is the optimal distribution of investments among the subsectors? and iii) How can we use these counterfactual results to guide future investment allocation?

Answering these questions is vital to understanding how the past distribution of investments might have contributed to some sluggishness in maximizing the benefits of the implemented investments. Nonetheless, this study has few limitations. Ideally, cost-benefit analysis remains the most informative method of determining the desirability of most types of investments in any given economy. However, data constraints imply a comprehensive cost-benefit analysis cannot always be carried out. In addition, this study evaluates the existing allocation of investments to maximize the positive change in GDP only in the short-run without considering the long-run effects of such distribution. Similarly, the study does not look into the linkages across sectors, the opportunity cost from not investing in one of the sectors and it is considered only a partial analysis where not all factors are taken into consideration. Finally, the study's findings do not include what should be called a risk/volatility index. This index should be included in any future studies.

The rest of the study is structured as follows: In Section 2, the mathematical models for the relation between investments and GDP growth is presented. For each sector of interest, we develop a stochastic differential equation (SDE) describing the ratio of change in GDP to investments. We also present the current situation where we use the data between 1982-2018. In Section 3, we present the optimization problem and the solution using the methods of stochastic control. In this section, we obtain an equation for the optimal distribution of investments among different sectors. Subsection 3.2 and 3.3 present the empirical results and the future forecast as well as the analysis of the findings. We include in subsection 3.4 trend analysis IMF (2019) to explain some of the findings. Section 4 is dedicated to our conclusions. The mathematical details are given in Appendix I.a. Appendix I.b. has parameter estimation equations. Appendix I.c. has a summary of the commonly used utility functions.

2. THE MATHEMATICAL MODEL

2.1 The Stochastic Differential Equation (SDE)

Assume that we have five sectors that are of interest to the economic planner: (1) agriculture, (2) manufacturing, (3) construction, (4) communications, and (5) tourism. The i^{th} "change in GDP/investments" is denoted as $S_i(t)$. The objective of the economic planner is to distribute his investments among the five sectors such that at the end of period T we have the maximum of some criterion.

All the variables are deflated, and the year 2002/2003 is the base year. The inflation rate, P(t), is modeled as GBM:

$$\frac{dP(t)}{P(t)} = \mu_P dt + \sigma_P dW_P(t) \qquad \text{II.1}$$

Where μ_P, σ_P, are unknown deterministic constants and $W_P(t)$ is a Wiener process.

Let us define the SDE of the stochastic process of the i[th] change in GDP/investments, $S_i(t)$, as:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW_i(t) \qquad \text{II.2}$$

$$S_i(0) = s_i > 0$$

where $W_i(t)$ is a Wiener process, μ_i is the expected level of the i[th] "change in GDP /investments", σ_i is the instantaneous intensity of fluctuations around μ_i. Equation (II. 2) has the solution:

$$S_i(t) = S_i(0)\exp\left[\mu_i\, t + \sigma_i\, W_i(t) - \frac{\sigma_i^2}{2} t\right] = S_i(0)\exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i\, \mathrm{W}_i(t)\right] \qquad \text{II.3}$$

i.e. $$\ln\left(\frac{S_i(t)}{S_i(0)}\right) = \left(\mu_i - \frac{\sigma_i^2}{2}\right)t + \sigma_i\, W_i(t) \qquad \text{II.4}$$

Define the volatility matrix σ as:

$$\sigma = \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & & \\ \dots & & \dots & \\ 0 & & & \sigma_n \end{bmatrix}, \text{ thus } \quad \sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_2} & & \\ \dots & & \dots & \\ 0 & & & \frac{1}{\sigma_n} \end{bmatrix} \qquad \text{II.5}$$

Assume that $\pi_i(t)$ is the money invested in the sector $S_i(t)$. The corresponding total change in GDP $X^\pi(t)$, with an initial value $x > 0$, is given as:

$$X^\pi(t) = \sum_{i=1}^{n} \pi_i(t) S_i(t) \qquad \text{II.6}$$

$$= \underline{\pi}^T(t)\underline{S}(t) \qquad \text{II.7}$$

Where $\underline{\pi}^T(t) = [\pi_1(t) \quad \pi_2(t) \quad \dots \quad \pi_n(t)]$, $\underline{S}^T(t) = [S_1(t) \quad S_2(t) \quad \dots \quad S_n(t)]$,

and $S(t) = \begin{bmatrix} S_1(t) & 0 & \dots & 0 \\ 0 & S_2(t) & & \\ \dots & & \dots & \\ 0 & & & S_n(t) \end{bmatrix}$.

Assuming slowly varying or almost constant shares $\pi_i(t)$ in the interval Δt we get:

$$dX^{\pi}(t) = \sum_{i=1}^{n} \pi_i(t) dS_i(t)$$

$$= \sum_{i=1}^{n} \pi_i(t) S_i(t) [\mu_i dt + \sigma_i dW_i(t)]$$

$$= \sum_{i=1}^{n} [\pi_i(t) S_i(t) \mu_i dt + \pi_i(t) S_i(t) \sigma_i dW_i(t)]$$

$$= \underline{\pi}^T(t) S \underline{\mu} dt + \underline{\pi}^T(t) S \sigma \, d\underline{W}(t) \qquad \text{II.8}$$

In terms of optimization, the above equation is equivalent to:

$$\frac{dX^{\pi}(t)}{X^{\pi}(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dS_i(t)}{S_i(t)}$$

$$= \sum_{i=1}^{n} \pi_i(t) [\mu_i dt + \sigma_i dW_i(t)]$$

$$= \underline{\pi}^T(t) [\underline{\mu} dt + \sigma \, d\underline{W}(t)] \qquad \text{II.9}$$

2.2 The Optimization Problem

The objective of the planner is to find the investment portfolio that maximizes the expected value of the utility of his change in GDP at time T; the end of the planning period. Specifically; we need to find

$$\max_{\underline{\pi}} E\{U(X^{\pi}(T))\} \qquad \text{III. 1}$$

Subject to the dynamic constraints:

$$dX^{\pi}(t) = X^{\pi}(t) [\underline{\pi}^T(t) \underline{\mu} dt + \underline{\pi}^T(t) \sigma \, d\underline{W}(t)] \qquad \text{II. 9}$$

$$X^{\pi}(0) = x$$

Define the market price of risk $\underline{\theta}(t)$ as:

$$\underline{\theta}^T(t) = \left(\sigma\sigma^T\right)^{-1}\sigma\underline{\mu}(t) \qquad \text{III. 2}$$

For square diagonal and invertible matrix σ, we get:

$$\underline{\theta}(t) = \sigma^{-1}\underline{\mu}(t) \qquad \text{III. 3}$$

The utility function is assumed to be of Constant Relative Risk Aversion (CRRA) form [see Appendix I.c]:

$$U(z) = \frac{z^{(1-\gamma)}}{(1-\gamma)} \qquad \gamma > 0 \qquad \text{III. 4}$$

After some manipulations (see appendix I.a), the optimal investments are given as:

$$\pi_i(t) = \left(\frac{1}{\gamma}\right)\left(\frac{\mu_i}{\sigma_i^2}\right)X^{\pi}(t) \qquad \text{A.14}$$

For a given pool of investments *Inv(t)*, the sectorial investments are given in terms of the mean and variance. Specifically;

$$\pi_i(t) = \frac{\left(\frac{\mu_i}{\sigma_i^2}\right)}{\sum_i\left(\frac{\mu_i}{\sigma_i^2}\right)}(Inv(t)) \qquad \text{A.15}$$

Since we have a GDP multiplier for each sector, m_i, we need to modify the investment plan, $\pi_i(t)$, to a new investment plan $\xi_i(t)$:

$$\xi_i(t) = \frac{m_i\left(\frac{\mu_i}{\sigma_i^2}\right)}{\sum_i m_i\left(\frac{\mu_i}{\sigma_i^2}\right)}(Inv(t)) \qquad \text{A.16}$$

This is the desired result.

2.3 The Portfolio Optimization

Each value of the variables was divided by the GDP deflator. The ratio of two GBM is a GBM. Thus, the deflated investments and the deflated change in GDP are both GBM. The ratio of the deflated change in GDP to deflated investments $S_i(t)$ is also a GBM and follows the equation:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW_i(t) \qquad \text{II.1}$$

2.4 Forecasting

Both investments and the change in sectoral GDP were modeled as GBM. The estimated drift, μ_i, and the diffusion coefficient, σ_i, were used in the forecasting. Specifically, we used only the drift value for the forecast till the year 2025. This yielded a trend curve. For more realistic situations, one should add the diffusion term. In this case, different forecasts for each simulation will be obtained. Instead we were content with the trend that represents the average of a large number of simulations.

2.5 The GDP Multiplier

Since the focus is on maximizing GDP, an important element is the GDP multiplier for each sector or subsector. For countries where multipliers are not readily available, general findings from the literature on other countries could be used. Specifically, the bucket approach (Batini, Eyraud, and Weber 2014), which is similar to the location quotient method, collect countries into three groups that are likely to have similar multiplier values based on their structural characteristics. The bucket method hypothesizes that similar factors affect multipliers in emerging economies (EMEs) and the least industrialized countries (LICs) where empirical and model-based estimates are not widely available and often of poor quality. For comparison, we use economic factors such as per capita GDP based on purchase power parity, debt to GDP ratio, and foreign trade pattern. For Egypt, the used bucket of countries include: (1) Greece, (2) Slovenia, (3) Czech Republic, (4) Estonia, and (5) Hungary. We used the average of their GDP multipliers for the different sectors. We obtained these values from the input/output table of each country as calculated in Abutaleb et. al (2013). Other countries could also be included. It should be noted that GDP multipliers change from year to year and from region to region.

3. Estimation Results

3.1 Total Investments in Egypt (1981/1982-2017/2018)

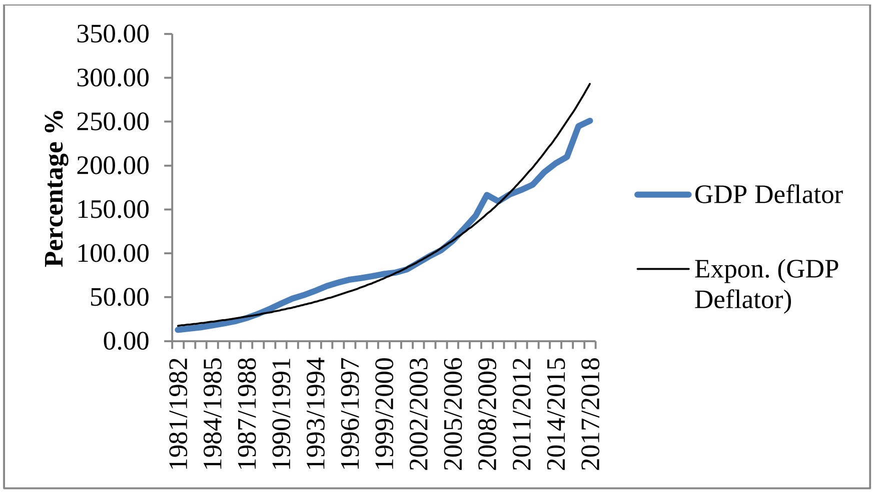
We normalize all financial values by the inflation deflator. The deflator follows a geometric Brownian motion (GBM) stochastic differential equation (SDE) and is shown in Figure 1. All the variables are deflated, and the year 2002/2003 is the base year. The inflation rate, P(t), is modeled as GBM:

$$\frac{dP(t)}{P(t)} = \mu_P dt + \sigma_P dW_P(t) \qquad \text{II.1}$$

Where μ_P, σ_P, are unknown deterministic constants and $W_P(t)$ is a Wiener process. The constants are estimated from the data and they were found to be: $\mu_P = 0.072$, $\sigma_P = 0.009$.

In Figure 1 (GDP deflator), year 2002/2003 is taken as numeraire (deflation is 100 percent). The vertical axis is the percentage deflation, and the horizontal axis is the year. We also show the trend curve.

Figure 1. GDP Deflator



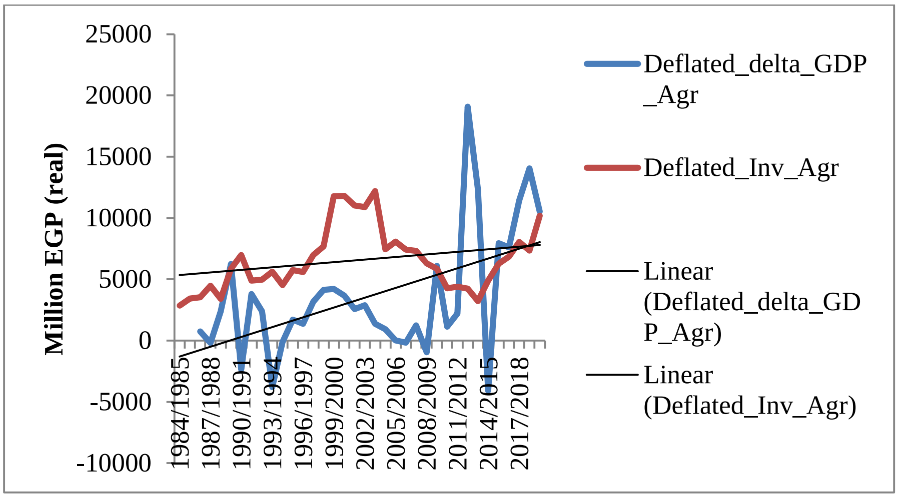
Source: Author's calculations based on data from the Ministry of Planning.

The focus of this study is the following sectors: (1) agriculture, (2) manufacturing, (3) construction, (4) communications, and (5) tourism. Manufacturing has around 23 subsectors such as coke and refining, food and beverages, basic metals, textiles, computers, paper, etc.

The deflated time series of investments among the different sectors is shown in Figures 2–6. In the same figures we show the deflated change in GDP and we have also plotted the trends in investments and the trends in the change in GDP. All the variables are deflated where the year 2002/2003 is the base year. The change in GDP and the investments in each sector are modeled as GBM. Dividing by the deflation rate, the deflated quantities are also GBM (see Appendix I.b).

In Figures 2–6, we show the deflated values of the change in GDP and the investments. Figure 2 shows that, for agriculture, the slope of the trend line in the deflated change in GDP is much bigger than that of the deflated investment. This means that small changes in investments will result in higher changes in GDP.

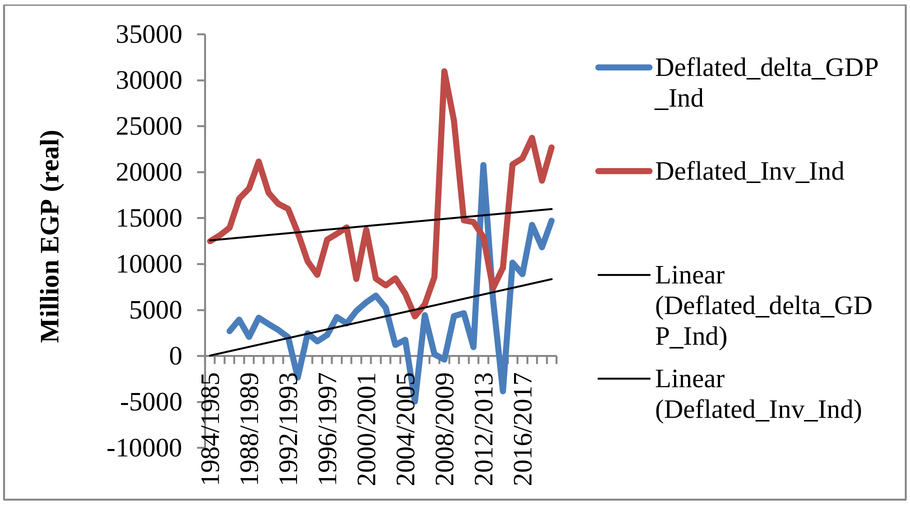
Figure 2. Agriculture, Deflated Values and Trends



Source: Author's calculations based on data from the Ministry of Planning.

Figure 3 shows that, for industry, the slope of the trend line in the deflated change in GDP is much bigger than that of the deflated investment. This means that small changes in investments will result in higher changes in GDP.

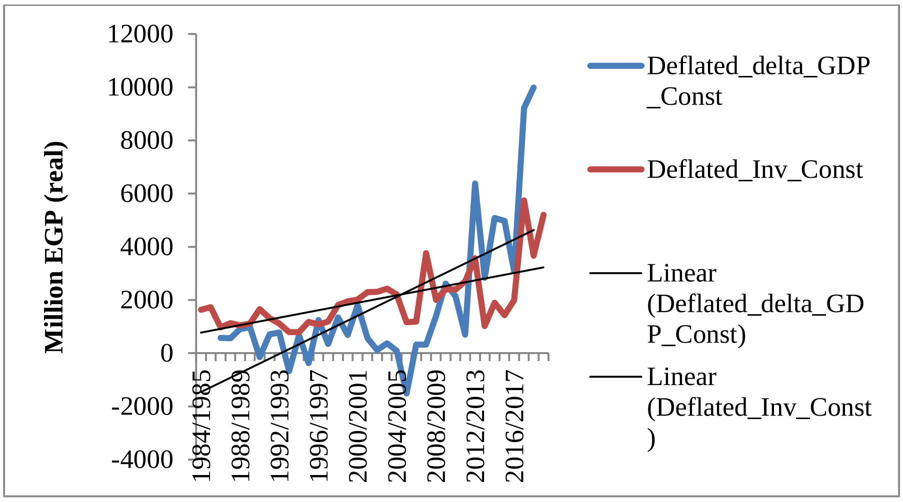
Figure 3. Industry, Deflated Values and Trends



Source: Author's calculations based on data from the Ministry of Planning.

Figure 4, for constructions, shows that the slope of the trend line in the deflated change in GDP is more than that of the deflated investment. This means that small changes in investments will result in higher changes in GDP.

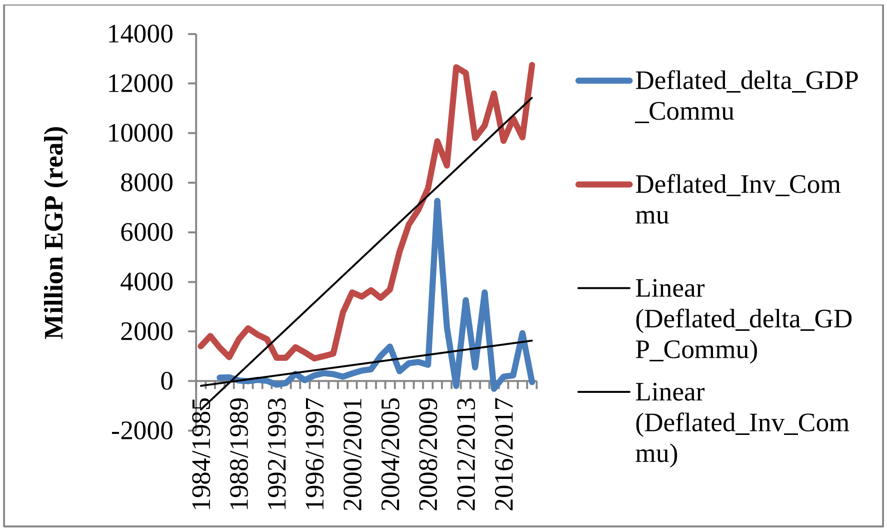
Figure 4. Construction, Deflated Values and Trends



Source: Author's calculations based on data from the Ministry of Planning.

Figure 5, for communications sector, the slope of the trend line in the deflated change in GDP is much less than that of the deflated investment. This means that the efficiency of investments is going down. Also big changes in investments will result in small changes in GDP.

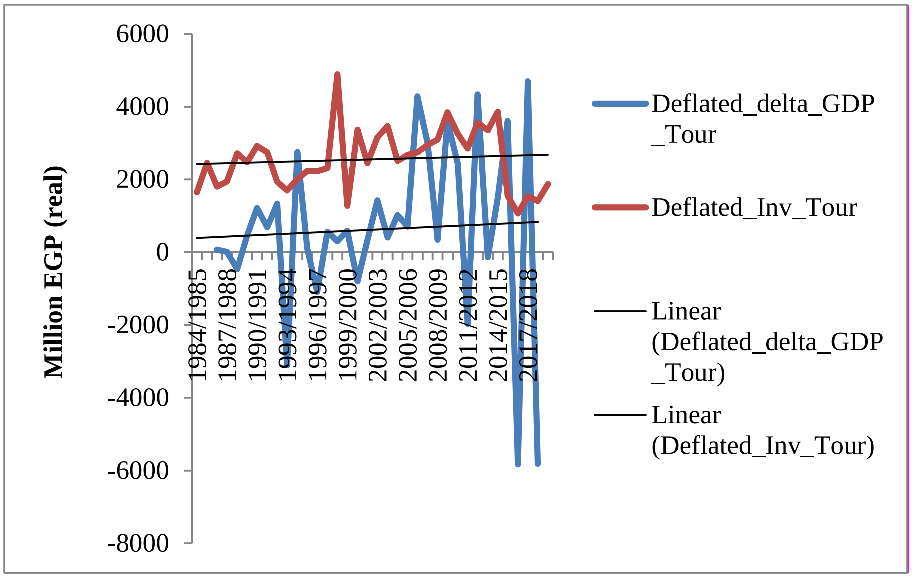
Figure 5. Communications, Deflated Values and Trends



Source: Author's calculations based on data from the Ministry of Planning.

Figure 6, for the tourism sector, shows that the slope of the trend line in the deflated change in GDP is almost the same as that of the deflated investment. This means that the efficiency of investments does not improve.

Figure 6. Tourism, Deflated Values and Trends



Source: Author's calculations based on data from Ministry of the Planning.

3.2 Overall Baseline Results: Optimal versus Actual Investments

In this study, the GDP multiplier was calculated by the bucket method (Abutaleb et al. 2013) and from the input output tables (Abdelsalam et al. 2015). In the bucket method, we used the average of the GDP multiplier of the above mentioned countries. The estimates are shown below in Table 1.

Table 1. GDP (output) Multiplier using Bucket method (Abutaleb) and I/O Tables (Abdelsalam), and the GDP Multiplier of Turkey

	GDP Multipier				
	Agriculture	Industry	Construction	Communications	Tourism
Abutaleb	1.460	4.000	4.930	3.100	2.260
Abdelsalam	1.390	2.440	5.230	1.690	2.900

The data of Turkey were obtained from D'Hernoncourt, Cordier, and Hadley (2011) and Atan and Arslanturk (2012).

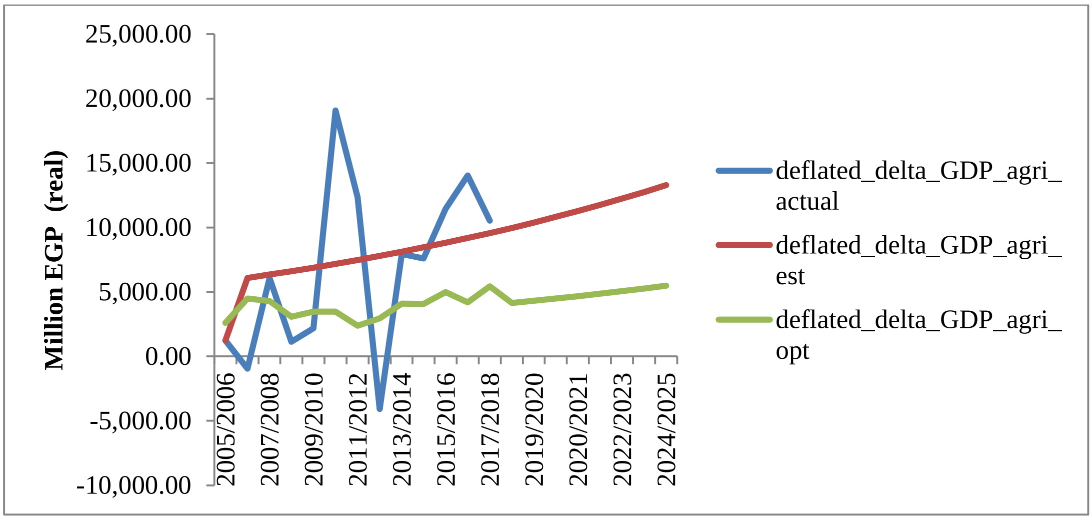
We used equation A. 16 to calculate the optimal sectorial investments $\xi_i(t)$ for each year starting 2005/2006. The actual total investments, Inv(t), was calculated, from the data, for each year till 2018 and forecasted onward using the GBM for investments. We used this value in equation A. 16 to find the optimal sectorial investments. We multiplied the calculated investments by the estimated ratio of the deflated change in GDP to deflated investments $S_i(t)$. This product, $\xi_i(t) * S_i(t)$, resulted in the optimal value of the deflated change in GDP. The optimal distribution of investments, before and after using the GDP multiplier, are shown in Table 2.

Table 2. Optimal Percentage Distribution (%) of Investments before and after Using the GDP Multiplier

Investment (%)	Agriculture	Manufacturing	Construction	Communications	Tourism
% without GDP Multiplier	24.190	27.020	20.970	11.290	16.530
% with GDP Multiplier	11.070	33.860	32.390	10.970	11.710

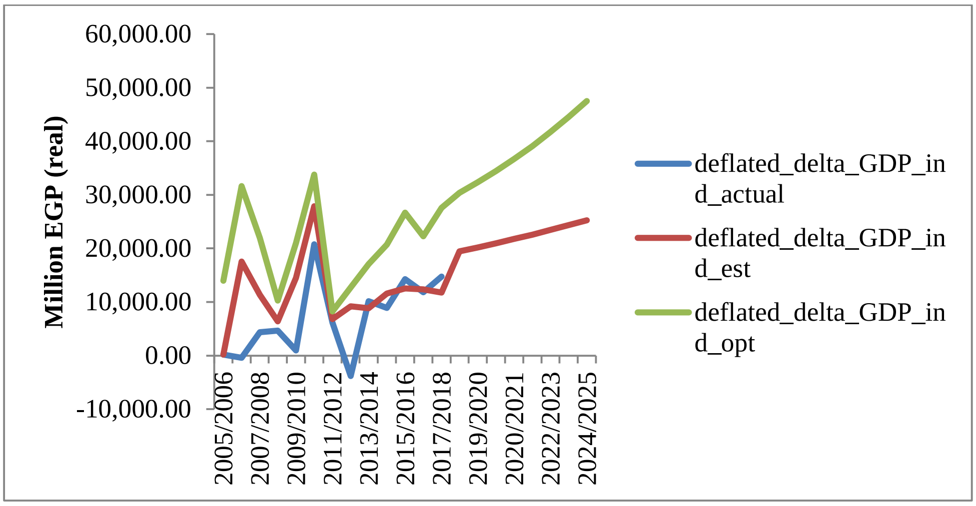
The actual and the estimated, using equation II. 1, and the optimal change in GDP, equation A. 16, are shown in Figures 7-12. The estimated values are averages since they should be randomly changing. We also present forecasts till the year 2024/2025.

Figure 7. Actual, Estimated, and Optimal Change in GDP, Agriculture



The optimal distribution of investments (green line) suggests reduction in the investments for the agriculture sector.

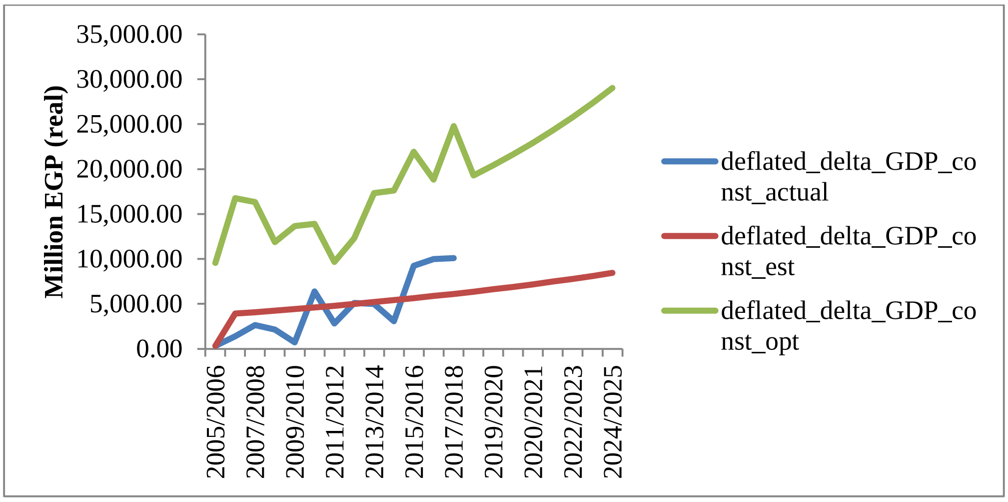
Figure 8. Actual, Estimated, and Optimal Change in GDP, Manufacturing



Source: Author's calculations based on data from the Ministry of Planning.

The optimal distribution of investments suggests an increase in the investments for the manufacturing sector. This would have resulted in almost 80 percent increase in the change in the GDP in 2017/2018. The forecast suggests an increase of almost 100 percent, in the year 2024/2025, using optimal strategy (green line) compared to the estimated values of the current strategy.

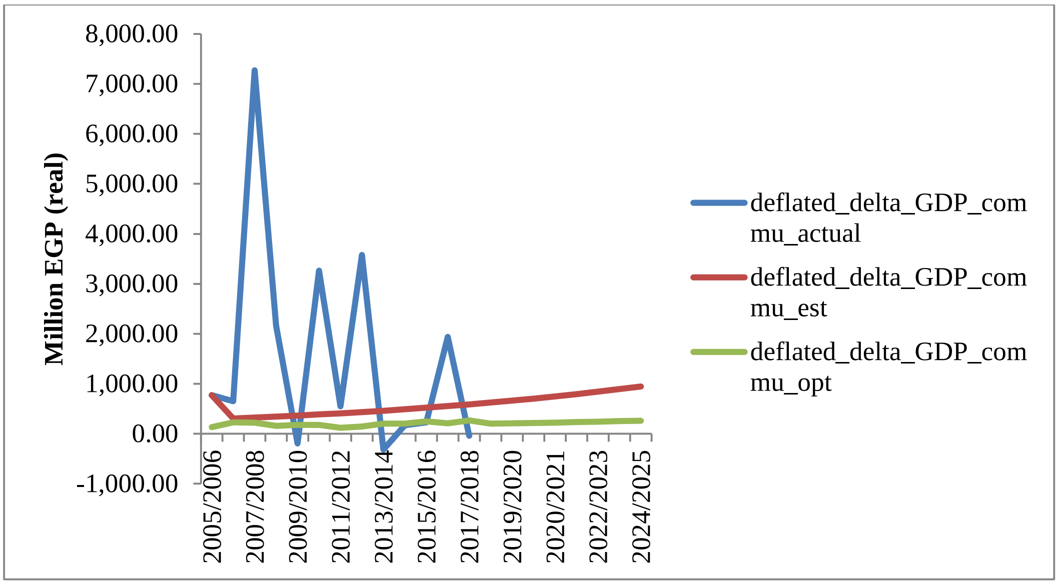
Figure 9. Actual, Estimated, and Optimal Change in GDP, Construction



The optimal distribution of investments suggests an increase in investments for the construction sector. This would have resulted in almost 100 percent increase in the change in GDP in 2017/2018. The forecast suggests an increase of almost 300 percent, in the year

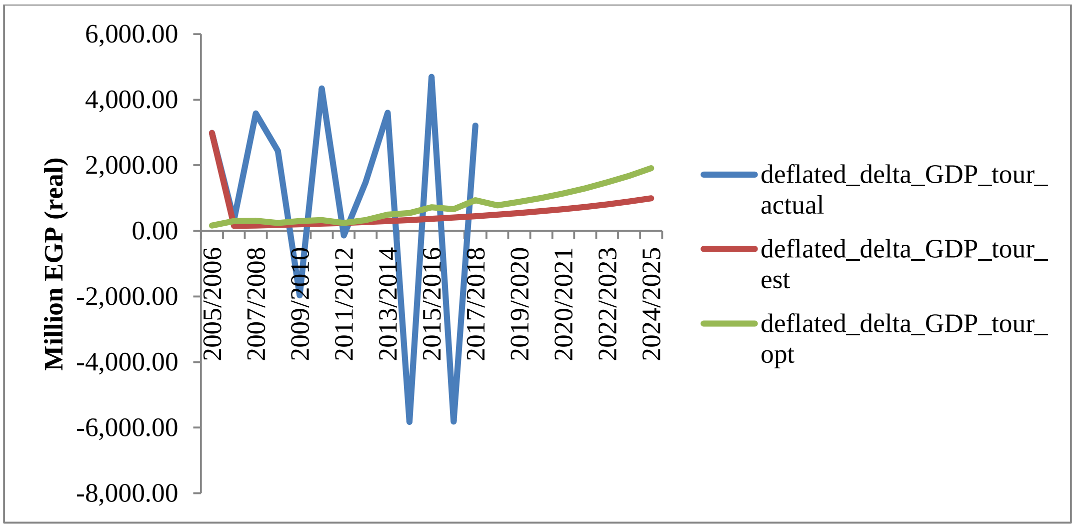
2024/2025, using optimal strategy (green line) compared to the estimated values of the current strategy.

Figure 10. Actual, Estimated, and Optimal Change in GDP, Communications



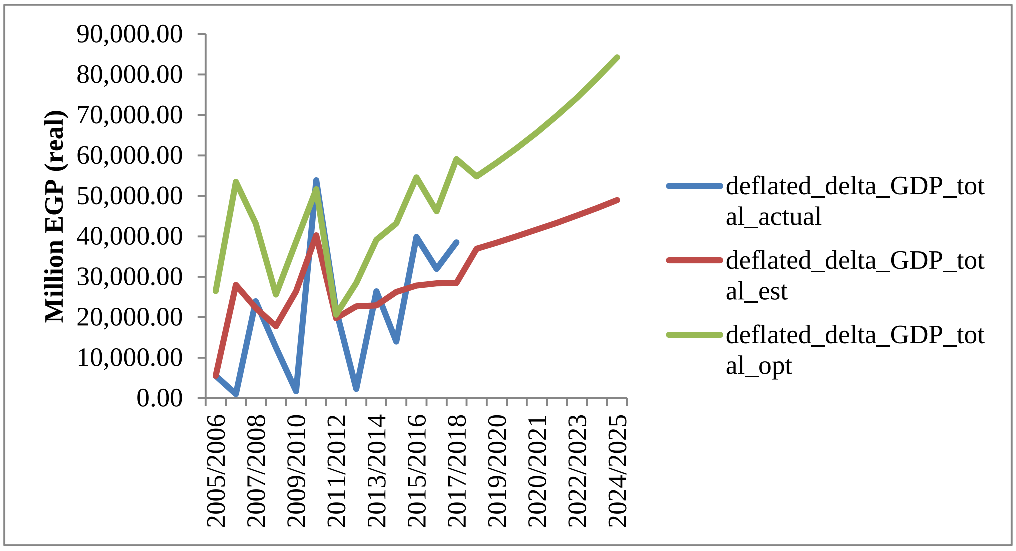
The optimal distribution of investments suggests a decrease in investments for the communications sector. This would have resulted in almost 50 percent decrease in the change in GDP in 2017/2018. The forecast suggests a decrease of almost 90 percent, in the year 2024/2025, using optimal strategy (green line) compared to the estimated values of the current strategy.

Figure 11. Actual, Estimated, and Optimal Change in GDP, Tourism



The optimal distribution of investments suggests an increase in investments for the tourism sector. This would have resulted in almost 100 percent increase in the change in GDP in 2024/2025 (green line).

Figure 12. Actual, Estimated, and Optimal Change in GDP, Total



The optimal distribution of investments would have resulted in almost 50 percent increase in the change in GDP in 2017/2018. The forecast suggests an increase of almost 80 percent, in the year 2024/2025, using optimal strategy (green line) compared to the estimated values of the current strategy (red line).

3.3 Manufacturing Analysis

The manufacturing sector has many subsectors. We have focused on the sectors with relatively high GDP. Both investments and the change in GDP followed the GBM model. We have estimated the difference in the average drift values (see Appendix I.b). If this difference is positive, this means that this subsector contributes in the economy more than it takes, i.e., it is a profitable subsector. We have sorted the subsectors according to this criterion using the data 1984-2011. The approximate results are (in descending order): (1) basic metal works that includes steel industry, (2) machinery, (3) tobacco, (4) fabricated metal, (5) computers, (6) food and beverages, (7) rubber and plastics, (8) pharmaceuticals, (9) petroleum refinery.

We have developed a scenario whereby, for manufacturing, all investments are directed towards the profitable subsectors of manufacturing such as basic metal works (which has a

GDP multiplier of 10.31). In this scenario, we keep the GDP multiplier unchanged for all the other sectors.

Table 3. Optimal Percentage Distribution (%) of Investments when Focusing on Profitable Subsectors of Industry Compared with Investing in all Subsectors

Investment (%)	Agriculture	Manufacturing	Construction	Communications	Tourism
% Profitable Subsectors of Industry	8.700	47.800	25.500	8.600	9.200
% with all Subsectors of Industry	11.000	33.800	32.400	10.970	11.710

In this table, we notice for the manufacturing sector that if we invest in the profitable subsectors we will increase investments to reach almost 48 percent of total investments. This is compared to almost 34 percent if we invest in all subsectors of manufacturing. All results are based on the optimal distribution of investments (equation A. 16).

Table 4. Total GDP (TOTGDP), Total Investments (TOTInv) in some of the Manufacturing Subsectors (nominal values in EGP Million) for Computer, Metal, Food, Chemicals, and Coke and Refinery

Year	1983	1994	2004	2011
TOTInv_Computer	14.54	45.75	66.79	285.40
TOTGDP_Computer	17.90	217.75	865.79	2,220.55
TOTInv_Metal	286.40	770.84	858.38	3,166.76
TOTGDP_Metal	348.83	3,662.88	13,223.44	34,160.92
TOTInv_Food	178.14	555.20	799.93	3,398.00
TOTGDP_Food	325.34	3,700.13	14,116.53	36,314.98
TOTInv_Chemicals	111.57	329.83	438.16	1,790.77
TOTGDP_Chemicals	140.64	1,636.63	6,335.49	16,280.65
TOTInv_Coke	673.65	1,651.56	1,451.61	4,402.27
TOTGDP_Coke	1,700.42	10,240.67	16,623.36	47,050.65

3.4 Trend Analysis

Using trend analysis (Munk 2012; Saplioglu 2015; IMF 2019), we drew trend lines in Figures 2-4. These trend lines are for both investments and the change in GDP. We then used the estimated, from the trend lines, values and obtained a scatter diagram between the $\delta\_GDP(t)$ and $Inv(t)$. It turned out that there is a linear relation between the variables. We then obtained this linear relation and it was found that the statistical R^2 was very close to 1 for the three sectors. The developed approximate relation between investments and the change in GDP for three sectors (agriculture, manufacturing, and construction) are given as follows:

(1) Agriculture: $\delta\_GDP(t) = -24{,}488.3 + 4.61\,Inv(t) + \varepsilon_A(t)$ IV.1

(2) Manufacturing: $\delta\_GDP(t) = -61{,}056 + 5.0\,Inv(t) + \varepsilon_I(t)$ IV.2

(3) Construction: $\delta\_GDP(t) = -4{,}257 + 3.7\,Inv(t) + \varepsilon_C(t)$ IV.3

Where $\delta\_GDP(t)$ is change in GDP and Inv(t) is Investments $\varepsilon_A(t)$, $\varepsilon_I(t)$, and $\varepsilon_C(t)$ are white Gaussian noise with zero mean and variance σ_A^2, σ_I^2, and σ_C^2, respectively.

These approximate equations are valid when we exceed the minimum values given by the negative numbers. For example, in manufacturing the minimum investments to reach a positive value is EGP Million12,000 .

Notice that, for agriculture, manufacturing and construction, the slope of investments is smaller compared to the slope of the change in GDP. The ratio of the slopes of change in GDP to investments, however, is higher for manufacturing (5.0) than construction (3.7) as shown in equations (IV. 1)-(IV. 3).

Using these equations and for the three sectors, we have constructed a table of the estimated change in GDP versus the estimated investments.

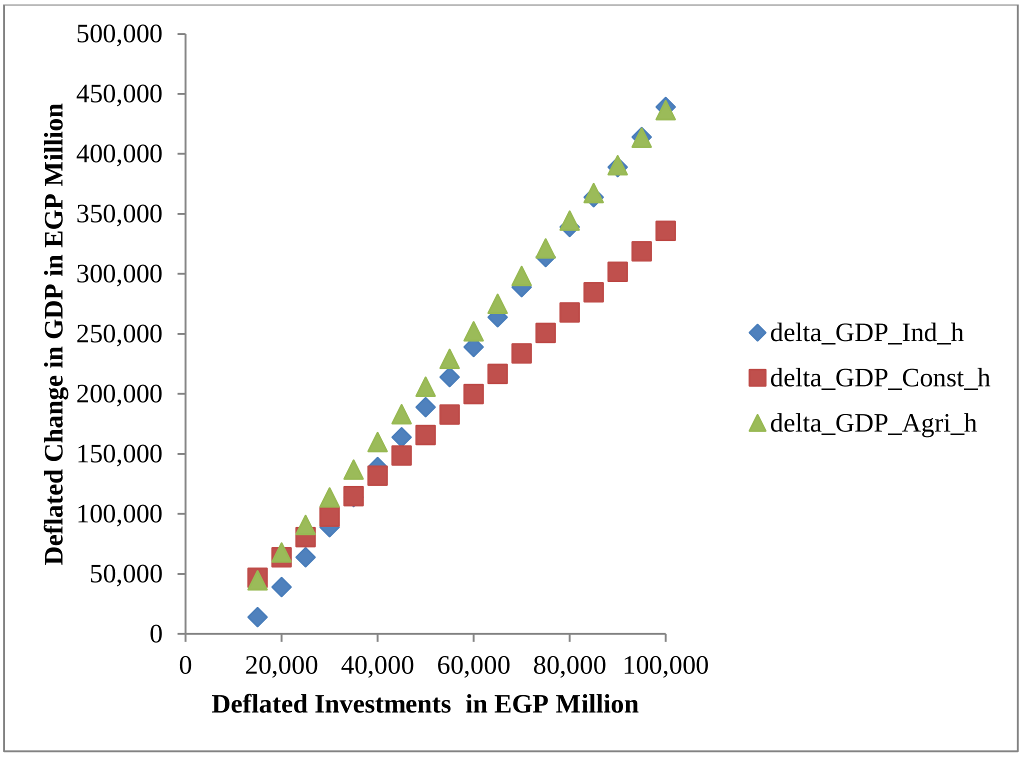
Table 5. The Change in GDP (delta_GDP_h) Produced by Change in Investments (real values in EGP Million) for Manufacturing, Construction, and Agriculture

Inv	delta_GDP_Ind_h	delta_GDP_Const_h	delta_GDP_Agri_h
15,000	13,899	46,743	44,662
20,000	38,884	63,743	67,712
25,000	63,869	80,743	90,762
30,000	88,854	97,743	113,812
35,000	113,839	114,743	136,862
40,000	138,824	131,743	159,912
45,000	163,809	148,743	182,962
50,000	188,794	165,743	206,012
55,000	213,779	182,743	229,062
60,000	238,764	199,743	252,112
65,000	263,749	216,743	275,162
70,000	288,734	233,743	298,212
75,000	313,719	250,743	321,262
80,000	338,704	267,743	344,312
85,000	363,689	284,743	367,362

90,000	388,674	301,743	390,412
95,000	413,659	318,743	413,462
100,000	438,644	335,743	436,512
105,000	463,629	352,743	459,562

In this table, investments (first column) less than EGP 35,000 mn will generate higher change in GDP for construction (third column) compared to manufacturing (second column). As we increase investments more than EGP 35,000 mn, the performance of the manufacturing sector outperforms the construction sector. As investments increase more than EGP 95,000 mn, the manufacturing sector ouotperforms both the agriculutre sector (fourth column) and the construction sector. The results are plotted in Figure 13.

Figure 13. Trend Analysis, Change in GDP versus Investments (Real Values)



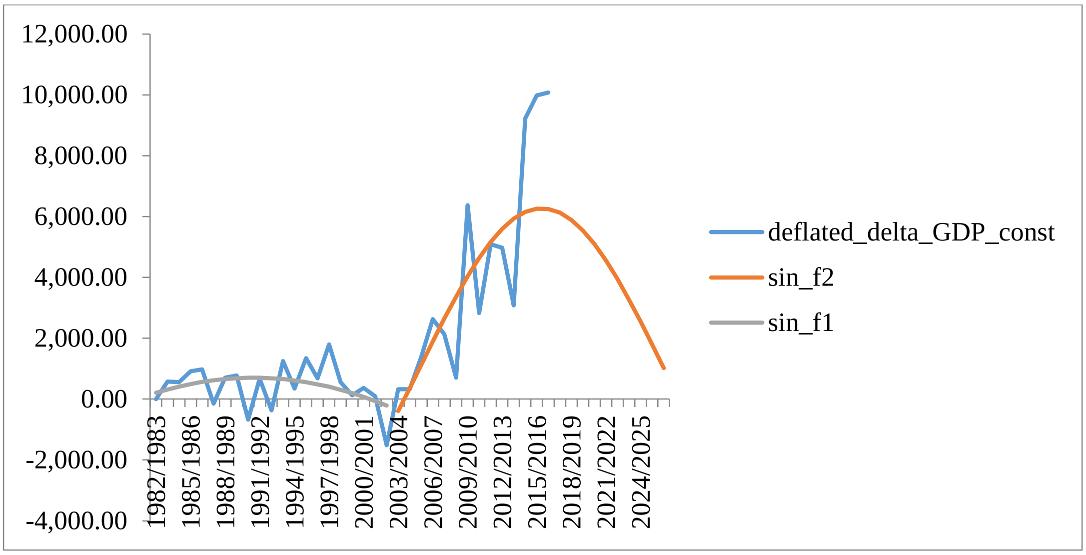
These results explain why in the classical portfolio approach, small increases in the construction investments resulted in larger increases in the change of the GDP compared to manufacturing. Because of that, the portfolio approach increases investments in construction relative to manufacturing. As the volume of investments increases, however, the manufacturing sector, for the same amount of investments, outperforms the other sectors including construction (see Table 4 and Figure 7-11, 13).

3.5 Senstivity Analysis

Sensitivity Analysis for the Agriculture, Manufacturing and Construction Sectors:

The three sectors are competing for investments. Trend analysis gives preference to agriculture and manufacturing. The GDP multiplier, however, shifts the investments away from agriculture and towards construction. The construction sector is volatile as we have seen in the 2008 global financial crisis (Jang et al. 2018). Eleven year ago, more precisely on September 15th, 2008, the financial crisis reached its peak with the insolvency of the investment bank Lehman Brothers. The trigger behind the worldwide collapse of the financial world was the burst of the housing bubble in the US. Heavy investments in the construction sector in Egypt might result in a similar crisis. In Figure 14, we show, for the construction sector, the real change of the GDP and the estimated cycles of change in GDP. The length of the cycle is around 20 years. Egypt suffered a real estate crisis in the year 2003/2004. It is estimated that the real estate values have reached their peaks and we are in the downturn. A real estate crisis might occur by the year 2024/2025.

Figure 14. Real Change in the Construction GDP, Estimated First Cycle (sin_f1), and Estimated Second Cycle (sin_f2)



The sustainability of GDP growth should rely on stable sectors such as manufacturing and agriculture. This could be reflected in a sustainability or risk index that should be developed, where the construction is given lower values compared to the other sectors. In Table 5, we reduce the construction GDP multiplier, while keeping the other sectors' multipliers unchanged, and we observe the impact on the distribution of investments. The change in the total GDP, for the different values of construction GDP multiplier, was small and thus it was not reported. This is because shifting investments to manufacturing, due to the change in the GDP multiplier, resulted in almost the same gain in the change of GDP. Notice that

$$\xi_i(t) = \frac{m_i\left(\frac{\mu_i}{\sigma_i^2}\right)}{\sum_i m_i\left(\frac{\mu_i}{\sigma_i^2}\right)}(Inv(t))$$ for manufacturing and construction are close. Thus, if we add something like risk index, the decision maker is better off investing in manufacturing than construction with almost the same change in the total GDP.

Table 6. Impact of Changing the Construction GDP Multiplier on Sectorial Investments

Construction GDP Multiplier	Investments in Agriculture %	Investments in Industry %	Investments in Constructions %
1	15%	45%	9%
2	14%	42%	16%
3	13%	39%	23%
4	12%	36%	28%
5	11%	34%	33%

As shown in the table, with construction GDP multiplier=1, the shift towards investments in manufacturing is clear; it is 45 percent compared to 9 percent for construction. The change in the total GDP for the different scenarios is small, less than 1 percent (not shown in the table). Thus, we could have obtained almost the same gain in GDP if we have directed our investments towards a stable sector such as manufacturing instead of the volatile construction sector. It should be mentioned that, according to the Asian Development Bank, Development Economics and Indicators Division (ERDI), Construction is classified as low technology industry since it heavily relies on steel and cement industries (Economic Indicators 2018).

Impact of the GDP Multiplier:

It shifts the investments towards sectors with high multiplier. The multiplier, however, is changing over time and sometimes it has negative values. This happened during the 2008 global crisis where the real estate sector was the force behind the collapse of several sectors of many economies. We recommend to continuously monitor the GDP multiplier using, for example, the bucket of countries approach. We next study the impact of the GDP multiplier on investments, and thus GDP, of the three major sectors: agriculture, manufacturing and construction.

Turkey GDP multipliers:

We made the same exercise using the GDP multipliers of Turkey (see Table 1). The results were different by almost 50 percent. The change in the GDP by the year 2017/2018 was almost

25 percent increase over the actual figures. This is much less than the optimal values using Abutaleb GDP multipliers that resulted in almost 50 percent increase in the year 2017/2018. This might be due to the fact Turkey GDP multipliers put almost equal weights for all sectors.

4. Conclusion and Policy Recommendations

Following the optimal distribution of investments in the given five sectors, and compared to the current pattern of investments, Egypt could have achieved an increase of almost 50 percent in the GDP of these sectors in the year 2017/2018 and 80 percent by the year 2024/2025. By changing the pattern of investments in the manufacturing subsectors, an increase of almost 100 percent could be achieved by the year 2024/2025 compared to the current pattern of investments. Actual data reveal that a positive change in investments yields a positive change in GDP. The ratio is more than one except for the communications sector. The maximum ratio is for the manufacturing sector. This means that we need to increase investments in the manufacturing sector and at a lesser extent in construction, and reduce investments in the communications sectors. The exact values of the redistribtution of investments were obtained in Sections 3. Notice that the trend analysis does not include volatility or the uncertainty in the estimated parameters. This is why, using optimal portfolio that includes uncertainities, the construction sector has a good portion of investments.

The decline in investments for the communications sector might be due to the fact that investments are in infrastructure, which is known to take around 10 years to show in the value added, i.e., it is a long-term investment. Also, if we direct investments towards the technology or the hardware sector of communications, the value added would increase.

As for the agriculture sector, the results are expected as we continue to invest in traditional crops such as wheat and corn. If we change the crops pattern to include food-processing-related-crops such as fruits, vegetables, and herbs, the GDP multiplier and the value added in the agriculture sector would increase for the same amount of investments.

In conclusion: (1) Stable sectors such as manufacturing should receive more investments than any other sector and investments should be higher than the current level. (2) Metal works, computers, and food industries should receive the highest portion of investments. (3) The GDP multiplier plays an important role in shifting investments between sectors. Thus, it is recommended that this multiplier should be reviewed on an annual basis. (4) We also recommend the inclusion of risk/sustainability index that should give more weights to manufacturing and agriculture.

Future work should include more sophisticated models for the economy not only GBM based models. The problem with more accurate models is the difficulty to find a closed form solution. Numerical methods should be utilized. This is currently under investigation.

APPENDICES TO PART I

Appendix I.a: The HJB equation for Optimal Distribution of Investments

In this appendix, we derive, using the Hamilton-Jacob-Bellman (HJB) equation, the optimal distribution of investments.

The HJB for the Optimization (Merton 1971, 1992; Abutaleb and Papaiouannou 2019):

The objective of the planner is to find the investment portfolio that maximizes the expected value of the utility of his total delta_GDP at time T, the end of the planning period. Specifically; we need to find

$$\max_{\underline{\pi}} E\{U(X^{\pi}(T))\} \quad \text{III.1}$$

Subject to the dynamic constraint:

$$dX^{\pi}(t) = \underline{\pi}^{T}(t) S \underline{\mu} dt + \underline{\pi}^{T}(t) S \sigma \, d\underline{W}(t) \quad \text{II.8}$$

with $X^{\pi}(0) = x$, $\pi_i(t)$ is the investments in the ith sector, and $\sum_{i=1}^{n} \pi_i(t) = Inv(t)$, where *Inv(t)* is the given total investments.

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dW_i(t) \quad \text{II.2}$$

In a matrix format, the system dynamics becomes:

$$\begin{bmatrix} dS_1 \\ dS_2 \\ \ldots \\ dS_n \\ dX^{\pi} \end{bmatrix} = \begin{bmatrix} \mu_1 S_1 \\ \mu_2 S_2 \\ \ldots \\ \mu_n S_n \\ \sum_{i=1}^{n} [\pi_i(t) S_i(t) \mu_i] \end{bmatrix} dt + \begin{bmatrix} \sigma_1 & 0 & \ldots & & 0 \\ 0 & \sigma_2 & 0 & \ldots & 0 \\ & & & & \\ & & & 0 & \sigma_n \\ \pi_1 S_1 \sigma_1 & \pi_2 S_2 \sigma_2 & \ldots & & \pi_n S_n \sigma_n \end{bmatrix} \begin{bmatrix} dW_1 \\ dW_2 \\ \ldots \\ \\ dW_n \end{bmatrix} \quad \text{A.1}$$

The utility function is assumed of Constant Relative Risk Aversion (CRRA) form (Munk 2012):

$$U(z) = \frac{z^{(1-\gamma)}}{(1-\gamma)} \qquad 0 < \gamma < 1 \qquad \text{A.2}$$

Or of HARA form:

$$U(z) = \frac{z^{(1-\beta)}}{(1-\beta)} \qquad 0 < \beta < \infty \qquad \text{A.3}$$

At instant "t", the objective function $\Psi(X^{\pi}, t)$ is defined as:

$$\Psi(X^{\pi}, T) = \max_{\underline{\pi}} E\{U(X^{\pi}(T))\} \qquad \text{A.4}$$

From the HJB equation of the optimal stochastic control we get:

$$0 = \frac{\partial \Psi}{\partial t} + \eta(S, \sigma, \underline{\mu}) + \max_{\underline{\pi}} \left\{ \mu^T S \underline{\pi}(t) \frac{\partial \Psi}{\partial X^{\pi}} + \frac{1}{2} tr\left[\frac{\partial^2 \Psi}{\partial X^{\pi^2}} \underline{\pi}^T(t) \sigma S S^T \sigma^T \underline{\pi}(t) \right] \right\} \qquad \text{A.5}$$

Where $\eta(S, \sigma, \underline{\mu})$ represents all the other terms independent of $\underline{\pi}(t)$. Thus, in the HJB equation A. 5 we have retained only the terms that are dependent on $\underline{\pi}(t)$. The maximization operation yields:

$$\max_{\underline{\pi}} \left\{ \mu^T S \pi(t) \frac{\partial \Psi}{\partial X^{\pi}} + \frac{1}{2} tr\left[\frac{\partial^2 \Psi}{\partial X^{\pi^2}} \pi^T(t) \sigma \sigma^T \pi(t) \right] \right\} = \max_{\underline{\pi}} \left\{ \frac{\partial \Psi}{\partial X^{\pi}} \sum_i \mu_i \pi_i + \frac{1}{2} \frac{\partial^2 \Psi}{\partial X^{\pi^2}} \sum_i (\sigma_i \pi_i)^2 \right\} \qquad \text{A.6}$$

which is reduced to:

$$\mu_i \frac{\partial \Psi}{\partial X^{\pi}} + \pi_i \sigma_i^2 \frac{\partial^2 \Psi}{\partial X^{\pi^2}} = 0 \qquad \text{A.7}$$

i.e., $$\pi_i = -\frac{\left(\mu_i \frac{\partial \Psi}{\partial X^{\pi}} \right)}{\left(\sigma_i^2 \frac{\partial^2 \Psi}{\partial X^{\pi^2}} \right)} \qquad \text{A.8}$$

Substitute inthe HJB equation, we get:

$$0=\frac{\partial\Psi}{\partial t}+\eta(S,\sigma,\underline{\mu})+\max_{\underline{\pi}}\left\{\frac{\partial\Psi}{\partial X^{\pi}}\sum_i\mu_i\pi_i+\frac{1}{2}\frac{\partial^2\Psi}{\partial X^{\pi^2}}\sum_i(\sigma_i\pi_i)^2\right\}$$

$$=\frac{\partial\Psi}{\partial t}+\eta(S,\sigma,\underline{\mu})-\frac{\partial\Psi}{\partial X^{\pi}}\sum_i\mu_i\frac{\left(\mu_i\frac{\partial\Psi}{\partial X^{\pi}}\right)}{\left(\sigma_i^2\frac{\partial^2\Psi}{\partial X^{\pi^2}}\right)}+\frac{1}{2}\frac{\partial^2\Psi}{\partial X^{\pi^2}}\sum_i\left(\sigma_i\frac{\left(\mu_i\frac{\partial\Psi}{\partial X^{\pi}}\right)}{\left(\sigma_i^2\frac{\partial^2\Psi}{\partial X^{\pi^2}}\right)}\right)^2 \quad \text{A.9}$$

which yields:

$$0=\frac{\partial\Psi}{\partial t}+\eta(S,\sigma,\underline{\mu})-\frac{1}{2}\frac{\left(\frac{\partial\Psi}{\partial X^{\pi}}\right)^2}{\left(\frac{\partial^2\Psi}{\partial X^{\pi^2}}\right)}\sum_i\left(\frac{\mu_i}{\sigma_i}\right)^2 \quad \text{A.10}$$

For $\Psi(X^{\pi},t)=g(t)(X^{\pi})^{\gamma}-t\eta(S,\sigma,\underline{\mu})$, we get:

$$0=\frac{\partial g}{\partial t}(X^{\pi})^{\gamma}-\frac{1}{2}\frac{\gamma^2g^2(t)(X^{\pi})^{2\gamma-2}}{\gamma(\gamma-1)(X^{\pi})^{\gamma-2}}\sum_i\left(\frac{\mu_i}{\sigma_i}\right)^2 \quad \text{A.11}$$

Which is reduced to:

$$0=\frac{\partial g}{\partial t}(X^{\pi})^{\gamma}-\frac{1}{2}\frac{\gamma^2g^2(t)(X^{\pi})^{\gamma}}{\gamma(\gamma-1)}\sum_i\left(\frac{\mu_i}{\sigma_i}\right)^2 \quad \text{A.12}$$

Eliminating $(X^{\pi})^{\gamma}$ we get for g(t) the ordinary differential equation:

$$\frac{\partial g}{\partial t}=\frac{1}{2}\frac{\gamma^2g^2(t)}{\gamma(\gamma-1)}\sum_i\left(\frac{\mu_i}{\sigma_i}\right)^2 \quad \text{A.13}$$

Substitute the expression of $\Psi(X^{\pi},t)$ in the control equation (A. 8) we get:

$$\pi_i(t)=-\frac{\left(\mu_i\frac{\partial\Psi}{\partial X^{\pi}}\right)}{\left(\sigma_i^2\frac{\partial^2\Psi}{\partial X^{\pi^2}}\right)}=-\frac{\left(\mu_i\gamma g(t)(X^{\pi})^{\gamma-1}\right)}{\left(\sigma_i^2\gamma(\gamma-1)g(t)(X^{\pi})^{\gamma-2}\right)}$$

$$= -\frac{\left(\mu_i\left(X^{\pi}\right)\right)}{\left(\sigma_i^2(\gamma-1)\right)} = \frac{\mu_i}{(1-\gamma)\sigma_i^2} X^{\pi}(t) \qquad \text{A.14}$$

$$= \frac{\mu_i}{(1-\gamma)\sigma_i^2} X^{\pi}(t)$$

For total investment at time "t" given by Inv(t), the investment in each sector becomes:

$$\pi_i(t) = Inv(t)\frac{\mu_i}{(1-\gamma)\sigma_i^2} X^{\pi}(t) / \sum_i \left[\frac{\mu_i}{(1-\gamma)\sigma_i^2} X^{\pi}(t)\right]$$

$$= Inv(t)\frac{\mu_i}{\sigma_i^2} / \sum_i \left(\frac{\mu_i}{\sigma_i^2}\right) \qquad \text{A.15}$$

This is the desired result.

The GDP Multiplier:

The GDP generated in any sector usually results in GDP in other sectors. We shall call this the GDP multiplier and is denoted as m_i for the ith sector. To include the GDP multiplier in our analysis, we use m_i as a weighting factor and we redistribute the investments $\xi_i(t)$ according to the equation:

$$\xi_i(t) = \frac{m_i\left(\frac{\mu_i}{\sigma_i^2}\right)}{\sum_i m_i\left(\frac{\mu_i}{\sigma_i^2}\right)} (Inv(t)) \qquad \text{A.16}$$

This is the desired result that was used in this study.

Appendix I.b: The GBM and Parameter Estimation

In this appendix, we present the derivations for the ratio of two GBM. We then present the maximum likelihood method to estimate the parameters of the GBM.

The SDE of the ratio of two GBM:

Let the SDE for the ith GBM be given by:

$$\frac{dP_i(t)}{P_i(t)} = \mu_i dt + \sigma_i dW_i(t)$$

Let $g(t, X)$ be twice continuously differentiable, and define

$$y(t) = g(t, X(t))$$

Then $$dy(t) = \frac{\partial g(t, X(t))}{\partial t} dt + \frac{\partial g(t, X(t))}{\partial X(t)} dX(t) + \frac{1}{2} \frac{\partial^2 g(t, X(t))}{\partial X^2(t)} dX^2(t)$$

Using Ito lemma, the SDE for $z_i(t) = \frac{1}{P_i(t)}$ is derived as:

$$dz_i(t) = \frac{-1}{P_i^2(t)} P_i\left(\mu_i dt + \sigma_i dW_i(t)\right) + \frac{1}{P_i^3(t)} P_i^2\left(\mu_i dt + \sigma_i dW_i(t)\right)^2$$

$$= -z_i\left(\mu_i dt + \sigma_i dW_i(t)\right) + z_i \sigma_i^2 dt$$

$$= z_i\left(\sigma_i^2 dt - \mu_i dt - \sigma_i dW_i(t)\right)$$

$$\frac{dz_i(t)}{z_i(t)} = \left(\sigma_i^2 - \mu_i\right)dt - \sigma_i dW_i(t)$$

We need the SDE of $y = P_1 / P_2 = P_1 z_2$. Using Ito lemma we get:

$$dy = d(P_1 / P_2) = P_1 dz_2 + dP_1 z_2 + dP_1 dz_2$$

$$= P_1 z_2\left[\left(\sigma_2^2 - \mu_2\right)dt - \sigma_2 dW_2(t)\right] + P_1 z_2\left[\mu_1 dt + \sigma_1 dW_1(t)\right] + P_1 z_2\left[\left(\sigma_2^2 - \mu_2\right)dt - \sigma_2 dW_2(t)\right]\left[\mu_1 dt + \sigma_1 dW_1(t)\right]$$

$$= P_1 z_2\left[\left(\sigma_2^2 + \mu_1 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_1 dW_1(t)\right]$$

$$= y\left[\left(\sigma_2^2 + \mu_1 - \mu_2\right)dt + \sigma_y dW_y\right]$$

Where $\sigma_y dW_y = -\sigma_2 dW_2(t) + \sigma_1 dW_1(t)$, and $\sigma_y = \sqrt{\sigma_1^2 + \sigma_2^2}$

Thus, $\frac{dy}{y} = (\mu_1 - \mu_2 + \sigma_2^2)dt + \sigma_y dW_y$, $y = P_1 / P_2 = P_1 z_2$

We know that:

$$P_1(t) = P_1(0)\exp\left[\mu_1\, t + \sigma_1\, W_1(t) - \frac{\sigma_1^2}{2}t\right] = P_1(0)\exp\left[\left(\mu_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1\, \mathrm{W}_1(t)\right]$$

i.e., $$\ln\left(\frac{P_1(t)}{P_1(0)}\right) = \left(\mu_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1\, W_1(t)$$

Similarly, $$\ln\left(\frac{P_2(t)}{P_2(0)}\right) = \left(\mu_2 - \frac{\sigma_2^2}{2}\right)t + \sigma_2\, W_2(t)$$

The deterministic ratio of $P_1(t)/P_2(t)$ is given by:

$$\frac{P_1(t)}{P_2(t)} = \frac{P_1(0)}{P_2(0)}\exp\left[\left(\mu_1 - \frac{\sigma_1^2}{2}\right)t + \sigma_1\, \mathrm{W}_1(t) - \left(\mu_2 - \frac{\sigma_2^2}{2}\right)t - \sigma_2\, \mathrm{W}_2(t)\right]$$

$$= \frac{P_1(0)}{P_2(0)}\exp\left[\left(\left(\mu_1 - \frac{\sigma_1^2}{2}\right) - \left(\mu_2 - \frac{\sigma_2^2}{2}\right)\right)t + \sigma_1\, \mathrm{W}_1(t) - \sigma_2\, \mathrm{W}_2(t)\right]$$

Define $dB(t) = \sigma_1\, \mathrm{dW}_1(t) - \sigma_2\, \mathrm{dW}_2(t)$, , $\mu_B = \mu_1 - \mu_2 + \sigma_2^2$, $\sigma_B = \sqrt{\sigma_1^2 + \sigma_2^2}$. Then the SDE of the ratio is given as:

$$\frac{d(P_1(t)/P_2(t))}{P_1(t)/P_2(t)} = \mu_B dt + \sigma_B dB(t)$$

The true solution of the ratio $\frac{dy}{y} = (\mu_1 - \mu_2 + \sigma_2^2)dt + \sigma_y dW_y$ is given by:

$$y(t) = \frac{P_1(t)}{P_2(t)} = \frac{P_1(0)}{P_2(0)}\exp\left[\left(\mu_1 - \mu_2 + \sigma_2^2 - \frac{\sigma_y^2}{2}\right)t + \sigma_y\, W_y(t)\right]$$

$$= \frac{P_1(t)}{P_2(t)} = \frac{P_1(0)}{P_2(0)}\exp\left[\left(\mu_1 - \mu_2 + \sigma_2^2 - \frac{(\sigma_1^2 + \sigma_2^2)}{2}\right)t + \sqrt{\sigma_1^2 + \sigma_2^2}\, W_y(t)\right]$$

$$= \frac{P_1(t)}{P_2(t)} = \frac{P_1(0)}{P_2(0)}\exp\left[\left(\mu_1 - \mu_2 - \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2}\right)t + \sqrt{\sigma_1^2 + \sigma_2^2}\, W_y(t)\right]$$

$$= \frac{P_1(t)}{P_2(t)} = \frac{P_1(0)}{P_2(0)} \exp\left[\left(\mu_1 - \frac{\sigma_1^2}{2}\right)t - \left(\mu_2 - \frac{\sigma_2^2}{2}\right)t + \sqrt{\sigma_1^2 + \sigma_2^2}\, W_y(t)\right]$$

Sometimes we need to divide two normalized values. Assume that we have $y(t) = \frac{P_1(t)}{P_2(t)}$ and $x(t) = \frac{P_3(t)}{P_2(t)}$ with SDE:

$$dy / y = \left[\left(\sigma_2^2 + \mu_1 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_1 dW_1(t)\right]$$

and
$$dx / x = \left[\left(\sigma_2^2 + \mu_3 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_3 dW_3(t)\right]$$

We need to find the SDE of z=x/y. Notice that both x and y have $\sigma_2 dW_2(t)$ as a common factor.

$$dz = xd(1/y) + (1/y)dx + dxd(1/y)$$

$$d(1/y) = \frac{-1}{y^2}dy + \frac{1}{y^3}(dy)^2$$

$$= \frac{-1}{y^2}y\left[\left(\sigma_2^2 + \mu_1 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_1 dW_1(t)\right] + \frac{1}{y^3}y^2\left[\left(\sigma_2^2 + \mu_1 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_1 dW_1(t)\right]^2$$

$$= \frac{-1}{y}\left[\left(\sigma_2^2 + \mu_1 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_1 dW_1(t)\right] + \frac{1}{y}\left(\sigma_1^2 + \sigma_2^2\right)dt$$

$$= \frac{-1}{y}\left[\left(\sigma_2^2 + \mu_1 - \mu_2 - \left(\sigma_1^2 + \sigma_2^2\right)\right)dt - \sigma_2 dW_2(t) + \sigma_1 dW_1(t)\right]$$

$$= \frac{-1}{y}\left[\left(\mu_1 - \mu_2 - \sigma_1^2\right)dt - \sigma_2 dW_2(t) + \sigma_1 dW_1(t)\right]$$

$$= \frac{1}{y}\left[\left(-\mu_1 + \mu_2 + \sigma_1^2\right)dt + \sigma_2 dW_2(t) - \sigma_1 dW_1(t)\right]$$

i.e.
$$d(1/y) = \frac{1}{y}\left[\left(-\mu_1 + \mu_2 + \sigma_1^2\right)dt + \sigma_2 dW_2(t) - \sigma_1 dW_1(t)\right]$$

Substitute into the equation $dz = xd(1/y) + (1/y)dx + dxd(1/y)$, we get:

$$dz = x\frac{1}{y}\left[\left(-\mu_1 + \mu_2 + \sigma_1^2\right)dt + \sigma_2 dW_2(t) - \sigma_1 dW_1(t)\right] + (1/y)x\left[\left(\sigma_2^2 + \mu_3 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_3 dW_3(t)\right]$$

$$+ x\left[\left(\sigma_2^2 + \mu_3 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_3 dW_3(t)\right]\frac{1}{y}\left[\left(-\mu_1 + \mu_2 + \sigma_1^2\right)dt + \sigma_2 dW_2(t) - \sigma_1 dW_1(t)\right]$$

$$dz = \left(\frac{x}{y}\right)\left[\left(-\mu_1 + \mu_2 + \sigma_1^2\right)dt + \sigma_2 dW_2(t) - \sigma_1 dW_1(t)\right] + \left(\frac{x}{y}\right)\left[\left(\sigma_2^2 + \mu_3 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_3 dW_3(t)\right]$$

$$-\left(\frac{x}{y}\right)\sigma_2^2 dt$$

Collecting terms we get:

$$dz = \left(\frac{x}{y}\right)\left[\left(-\mu_1 + \mu_2 + \sigma_1^2\right)dt + \left(\sigma_2^2 + \mu_3 - \mu_2\right)dt - \sigma_2^2 dt - \sigma_1 dW_1(t) + \sigma_3 dW_3(t)\right]$$

i.e., $dz = z[(-\mu_1 + \mu_3 + \sigma_1^2)dt - \sigma_1 dW_1(t) + \sigma_3 dW_3(t)]$

Thus, we only need to know the mean and variance of the original values $P_1(t)$ and $P_3(t)$. $P_3(t)$ could be nominal delta_GDP and $P_1(t)$ could be nominal investments. In this case, $z = \left(\frac{P_3}{P_1}\right)$ and we do not need to know the deflator. If $\left(-\mu_1 + \mu_3 + \sigma_1^2\right) > 0$, then this sector is profitable.

Remember that $z = \left(\frac{x}{y}\right)$, $y(t) = \frac{P_1(t)}{P_2(t)}$ and $x(t) = \frac{P_3(t)}{P_2(t)}$:

$$dy/y = \left[\left(\sigma_2^2 + \mu_1 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_1 dW_1(t)\right]$$

and $$dx/x = \left[\left(\sigma_2^2 + \mu_3 - \mu_2\right)dt - \sigma_2 dW_2(t) + \sigma_3 dW_3(t)\right]$$

Parameter Estimation:

For a stochastic process that is a GBM, we need to estimate the mean and the variance. Let the process be defined as:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t)$$

In this case:

$$d\ln\left(S(t)\right)=\left(\mu-\frac{1}{2}\sigma^2\right)dt+\sigma dW(t)$$

Define $r(t)=\ln\left(S(t)\right)-\ln\left(S(t-1)\right)=\ln\left(\frac{S(t)}{S(t-1)}\right)$

Then r(t) has a Normal distribution with $E\{r(t)\}=\left(\mu-\frac{1}{2}\sigma^2\right)$ and variance $E\left\{\left[r(t)-E\{r(t)\}\right]^2\right\}=\sigma^2$. These are the equations that we will use to estimate the unknown parameters of the GBM σ^2 and μ.

Appendix I.c: Utility Functions

In this appendix, we show some of the commonly used utility functions for the analysis of economic and financial data (Karatzas 1997). It is assumed that the investor is risk averse. Therefore, his utility function must be concave.

Definition C.1: In the economic literature, a concave utility function, u(.), is often referred to a continuous function u: $(0,\infty) \rightarrow R$ which is (strictly) increasing, (strictly) concave, continuously differentiable. More rigorously, a (concave) utility function should also satisfy the Inada conditions that

$$\left.\frac{du(z)}{dz}\right|_{z=0+} = \infty$$

$$\lim_{z\to\infty} \frac{du(z)}{dz} = 0$$

A risk-loving investor should have a convex utility function while a risk-neutral investor should have a linear utility function. In the definition above, the requirement that a utility function be (strictly) increasing says that an increase in z (z can be, for example, consumption or wealth) increases the utility; the (strict) concavity implies a diminishing marginal utility, that is, the utility gain decreases with an increase of z. The infinite marginal utility when z approaches the origin implies that 'something is much better than nothing' and the vanishing marginal utility when z approaches ∞ suggests that, for an extremely rich investor, the utility gain from a small increase of wealth or consumption can be ignored.

A concave utility is associated with a risk-averse investor and the degree of curvature of the corresponding utility function determines the intensity of the investor's risk aversion. Curvature can be measured by the second derivative of the utility function, scaled by the first derivative. There are two main measures of risk aversion in economics. One is the absolute risk aversion (ARA), which is defined by:

$$\text{ARA(z)} \equiv - \frac{d^2u(z)}{dz^2} / \frac{du(z)}{dz}$$

The other measure is the relative risk aversion (RRA) defined by

$$\text{RRA(z)} \equiv -\text{z}\frac{d^2u(z)}{dz^2} / \frac{du(z)}{dz}$$

Frequently used utility functions:

The following utility functions appear to be frequently used in the literature of economics and finance. Each has its own attractive and unattractive features.

(i) Quadratic Utility Function

u(z) = $az - bz^2$, a ≥ 0, b > 0 and 0 < z < (a /2b)

A quadratic utility function can make an optimization model more tractable, in particular, when uncertainty is involved. This is due to its characterization of linear marginal utility. However, quadratic utility is an implausible description of behavior toward risk as it implies an increasing absolute risk aversion in z. It is a common thought that absolute risk aversion should decrease, or at least should not increase with z. Moreover, this utility function does not satisfy the Inada conditions.

(ii) CARA-Exponential Utility Function

u(z) = $-e^{-\gamma z}$, γ > 0 and z > 0

The exponential utility function is known as a constant absolute risk aversion, or CARA in short, its absolute risk aversion is constant and equal to γ. Exponential utility can produce simple results if asset returns are normally distributed. The shortcoming of this function is that it implies negative consumption or wealth which is not desirable in most cases. This utility function satisfies the Inada condition (ii) but violates the Inada condition (i).

(iii) CRRA-Power Utility Function

u(z) = $\frac{z^{1-\gamma}}{1-\gamma}$, γ > 0, $\gamma \neq 1$ and z > 0

The power utility has a constant relative risk aversion of γ, and whence CRRA. This utility implies that the absolute risk aversion is declining in z and excludes negative consumption or wealth. The power utility function can produce simple results when asset returns are log normally distributed. Furthermore, it satisfies both Inada conditions (i) and (ii). These are perhaps the main reasons why the CRRA utility function is so commonly employed in the literature. The coefficient 1/γ is referred to as the elasticity of substitution of consumption in economics. From the discussion above, it appears that the CRRA utility is the most reasonable description of an investor’s aversion to risk. Therefore, we will focus on the CRRA utility.

PART II: AN APPLICATION OF THE STOCHASTIC CONTROL APPROACH TO: HEDGING SUDDEN STOPS AND PRECAUTIONARY RECESSIONS FOR EGYPT'

1. INTRODUCTION AND LITERATURE REVIEW:

Developing countries' reserves have increased dramatically in recent years; growing by more than 60 percent since the Asian financial crisis of 1997 (Mendoza 2004). Some policymakers argue that this is the best practice to insure against future crisis. The implications of such self-insuring strategy are that a significant level of deadweight losses would nonetheless be incurred (Calvo 1998; Calvo, Izquierdo, and Mejia 2004). Despite the ongoing debate on reserve issues, there is little consensus about how to assess reserve holdings in different economies, even though this is an important aspect of any country external stability assessment (IMF 2015).

We define financial crisis and crashes as "Sudden Stops." Even well managed developed economies suffer from sudden stops or crashes. The recent 2008 global financial crisis is a proof of this. The situation is even worse for emerging economies such as that of Egypt. Foreign direct investments (FDI) are needed to finance new long-term projects, especially in emerging markets where capital is hard to find. In the case of crash, at a moment's notice these economies are required to face capital outflows as happened with Egypt in 2011. This has serious repercussions on the economy and might very well trigger a recession. In a typical sudden stop, external funding declines by 10 percent or more, and the main impact lasts for over six years.

In the past few years, there has been considerable effort to identify guidance on the appropriate level of reserves for less mature market economies such as that of Egypt. The traditional rules for reserves have the attraction of being relatively intuitive, and simple, yet at the same time they are partial and narrow in scope. The rules include:

(1) For countries with less open capital accounts, three months' coverage of imports is typically used as a benchmark.
(2) The "Greenspan-Guidotti" rule of 100 percent cover of short-term debt—is the most widely-used standard.
(3) For countries with large banking sector and very open capital accounts, the ratio of reserves to broad money (typically M2) is typically set at 20 percent which is about 5 percent of GDP.

(4) The expanded Greenspan-Guidotti rule, consisting of short-term debt plus the current account deficit, which is intended to reflect the full potential 12-month financing need.

While there is a substantial agreement on the kind of policy adjustments that reduce domestic risk, there is less consensus on the adequate external liability management strategies to deal with external shocks or sudden stops. More recently, optimal reserve models were developed to integrate cost and benefit considerations. A widely used model is that of Jeanne and Rancière (2006), where the optimal level of reserves is determined by balancing the economic cost (the potential loss in output and consumption, given the size and probability of the sudden stop) with the opportunity cost of holding reserves, and reflecting the degree of risk aversion. An issue with this approach is that it can result in a wide range of estimated optimal reserve holdings, depending upon its calibration

Recently, it has been suggested that emerging economies should reduce non-contingent reserve and replace it with what is known as contingent reserve (Caballero and Panageas 2003, 2004). The contingent reserve is a set of contracts in the stock market that move with what is known as the volatility index (VIX). This VIX has proven to be highly correlated with the appearance of sudden stops. The cost of buying such contracts is almost 10 percent or less of the cost of having reserves. Once a sudden stop is detected and the VIX increases, the contracts are due and their amount is used to offset the loss in the current account of emerging economies (Abutaleb and Gaber 2012).

In this report, we present exact expressions for the optimal consumption and the optimal reserves. We also derive an approximate relation between consumption and reserves. It turned out that this suboptimal relation mimics, to a great extent, the consumption and reserves pattern of Egypt.

Treating the reserves as assets with risk-free interest, we were able to derive an exact expression for the optimal reserves and the optimal consumption. The objective function or the optimality criterion was the discounted utility of the normalized consumption. The normalization was with respect to GDP. The used utility function was constant relative risk aversion (CRRA) with a parameter “ γ ” reflecting the degree of risk aversion (Munk 2012). In this approach, we are not concerned about the usage of the reserves; it is simply an optimization problem. The martingale optimality principle was used for the derivations.

2. SUDDEN STOPS AND THE MATHEMATICAL MODEL:

We study a representative agent economy with a responsible government that seeks to maximize the expected present value of the utility from consumption C(t):

$$E\left\{\int_t^{\infty} U\big(C(s)\big)e^{-r(s-t)}ds\right\} \qquad , \text{t < s} \qquad \text{(II.1)}$$

where r is the riskless interest rate and the discount factor. The utility function U(C(t)) has many shapes, and in this analysis, we shall use the CRRA shape given as (Munk 2010 NOT LISTED, PLEASE LIST OR DELETE)]:

$$U(C) = \frac{C^{1-\gamma}}{1-\gamma} \quad , \ \gamma > 0, \gamma \neq 1 \qquad \text{(II.2)}$$

2.1 Emerging Market Economies and World Capital Markets:

This section presents some evidence on the behavior of domestic absorption, output, and reserves in emerging market economies experiencing sudden stops in capital flows.

Let Y(t) represent the country's income (GDP) in the pre-development phase, and assume that it follows the Geometric Brownian motion model. The stochastic differential equation (SDE) of Y(t) is given as:

$$dY(t) = \mu_Y Y(t)dt + \sigma_Y Y(t)dB(t) \qquad \text{II.3}$$

where μ_Y is the growth rate of GDP, σ_Y is the market volatility and B(t) is the Brownian motion or the Wiener process.

Y(t) SDE has the solution:

$$\ln\left(\frac{Y(t)}{Y(0)}\right) = \left(\mu_Y - \frac{\sigma_Y^2}{2}\right)t + \sigma_Y\ B(t)$$

Expressing all variables in terms of current US dollar we get: $\left(\mu_Y - \frac{\sigma_Y^2}{2}\right)$=0.07724,

Using the maximum likelihood method, we get: $\mu_Y = 0.08$, $\sigma_Y^2 = 0.0062$

A country in its developing stage would like to borrow against its post development income. The potential financiers are: (1) World capital markets (WCM), and (2) Specialists. Unlike the specialist, WCM have limited understanding of emerging markets hence they do not accept contracts that are related to the emerging economies. The country can also accumulate international assets X(t). Both assets and liabilities pay a return of r per unit of time.

2.2 Specialists and Sudden Stops:

Specialists are investors who are familiar with the emerging markets at large and are willing to invest in many areas where the WCM will not invest. In practice, they have equity investments, FDI, the riskiest tranches of GDP-indexed bonds, or toxic-waste more generally. Thus, during non-sudden stop times "NSS" or normal times, the maximum flow of resources received is "$\bar{f}Y(t)$":

$$\max f^{NSS}(t) = \bar{f} \qquad \text{II.4}$$

During sudden stops, the maximum flow of resources received from the specialists is "$\underline{f}Y(t)$" with:

$$\underline{f} < \bar{f} \qquad \text{II.5}$$

Thus

$$f^{SS}(t) = \underline{f} \qquad \text{II.6}$$

Define A(t) as the sum of income and contingent flows from specialists:

$$A(t) = \left(\theta^{NSS} 1\{NSS\} + \theta^{SS} 1\{SS\}\right) Y(t) \qquad \text{II.7}$$

where $\theta^{NSS}(t) \leq (1 + \bar{f})$ II.8

$$\theta^{SS}(t) \leq (1 + \underline{f}) \qquad \text{II.9}$$

$$1\{NSS\} = \begin{cases} 1 & \text{Country in normal times} \\ 0 & \text{elsewhere} \end{cases} \qquad \text{II.10}$$

$$1\{SS\} = \begin{cases} 1 & \text{Country in sudden stop times} \\ 0 & \text{elsewhere} \end{cases} \qquad \text{II.11}$$

Note that $\theta^{SS} < \theta^{NSS}$

The net assets accumulation or reserves X(t) is now described by:

$$dX(t) = \left[rX(t) - C(t) + A(t)\right]dt \qquad \text{II.12}$$

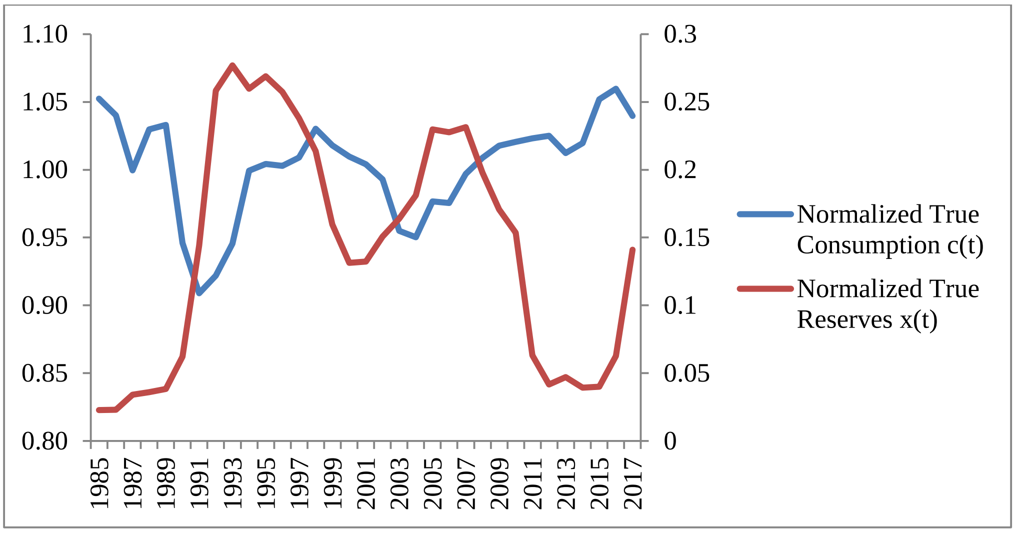
$$A(t) = Y(t) + "net\_foreign\_accumulation"$$

The change in reserves is due to: (1) Interest on reserves at r%, (2) The difference between the "GDP + net foreign accumulation," A(t), and the consumption C(t). The consumption, C(t), is defined as the sum of government consumption, household consumption, plus private investment. C(t) also equals to GDP +Current account balance. Note that equation (II. 12) shows a negative relationship between international reserves, X(t), and the domestic absorption or the consumption C(t).

2.3 The case of Egypt:

We define the normalized variables $c(t) = C(t)/(\psi Y(t))$ and $x(t) = X(t)/(\psi Y(t))$, where ψ is a constant value that could be 1 or other values. It was noticed that the relation between c(t) and *x*(t) follows what is known as a prey-predator equation. As c(t) increases *x*(t) decreases and then c(t) decreases and *x*(t) increases in a cyclic behavior. We have calculated the cycle, using Fourier transform, to be around 7 years. This is shown in Figure 1, where we present the data for 1985-2017.

Figure 1. Normalized True Consumption and Reserves for Egypt



2.4 The Optimization Problem:

In what follows, we shall study the problem for an economy facing temporary shocks (sudden stops). We shall study the case where the economy is normal, i.e., no sudden stops (NSS).

Define $$V\big(X(0),Y(0)\big)=\max_{C(s)} E\left\{\int_0^{\tau^{SS}} e^{-rs}U(C(s))ds+\varphi\big(X(\tau^{SS}),Y(\tau^{SS})\big)\right\}$$

= value function in the normal state II.13

$\varphi\big(X(\tau^{SS}),Y(\tau^{SS})\big)$=Utility of the desired value of the Reserves at the onset of SS II.14

τ^{SS} = time to sudden stop which is random

$$U(C)=\frac{C^{1-\gamma}}{1-\gamma} \qquad \gamma>0,\gamma\neq 1 \qquad \text{II.15}$$

Transition from normal times to sudden stops occurs with a constant hazard rate λ at the random time τ^{SS}. The constraints and the system dynamics are:

$$dY(t)=\mu_Y Y(t)dt+\sigma_Y Y(t)dB(t) \qquad \text{II.3}$$

$$A(t)=\theta^{NSS}Y(t) \qquad \text{II.11}$$

θ^{NSS} for developing countries is around 1.2 (Caballero and Panageas 2004).

It turned out that θ^{NSS} for Egypt is around 1 and is changing over time. It was taken to be A(t)/Y(t).

$$dX(t)=\left[rX(t)-C(t)+A(t)\right]dt \qquad \text{II.12}$$

$$X(t)\geq 0$$

In vector format, equations (II. 3) , (II. 11) and (II. 12) are written compactly as:

$$\begin{bmatrix} dX(t) \\ dY(t) \end{bmatrix}=\left\{\begin{bmatrix} r & \theta^{NSS} \\ 0 & \mu_Y \end{bmatrix}\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}+\begin{bmatrix} -C(t) \\ 0 \end{bmatrix}\right\}dt+\begin{bmatrix} 0 \\ \sigma_Y Y(t) \end{bmatrix}dB(t) \qquad \text{II.17}$$

The country is faced with the decision of how much to consume $C(t)$ in order to maximize the utility function and at the same time ending with a desired level of reserves $X(\tau^{SS})$ with a utility function $\varphi\big(X(\tau^{SS}),Y(\tau^{SS})\big)$. The final value of reserves $X(\tau^{SS})$ could be defined by the decision maker or optimally estimated. The reserves, $X(t)$, play the role of providing the country with resources during sudden stops. Accumulating reserves, however, is costly and deprives the economy from precious resources that could be used for development.

The optimization problem is to find the optimal consumption path and at the same time satisfying the level of the desired reserves at the end time of sudden stop τ^{SS}. After normalization w.r.t. $\psi Y(t)$ we get (see Appendix II.a):

$$dx(t) = \left[x(t)\left(r - \mu_Y + \sigma_Y^2\right) + \left(\left(\frac{\theta^{NSS}}{\psi} \right) - c(t) \right) \right] dt - x(t)\sigma_Y dB(t) \qquad \text{A.3}$$

where $c(t) = C(t)/\psi Y(t)$ and $x(t) = X(t)/\psi Y(t)$

An approximate solution is obtained by assuming time-invariance of the objective function, the optimal normalized consumption $c(x)$ as function of the normalized x(t) reserves was derived as (see Appendix II.a):

$$c(x) = K_1 + K_2 \frac{x^{\frac{2}{\sigma_Y \gamma}}}{1 - x^{\frac{2}{\sigma_Y \gamma}}} \qquad \text{A.8}$$

where K_1 and K_2 are constants to be determined from the boundary conditions or by minimizing the sum of squared error between the observed c(t) and the estimated c of eqn. (A. 8) and using the observed x(t). The relationship between c(t) and x(t) of Equation A. 8 is close to the prey-predator model. This means that the approximate optimal solution reflects what we observe in reality.

An approximate SDE of the normalized reserves becomes:

$$dx(t) = x(t)\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{1}{x(t)} \left(K_1 + K_2 \frac{x^{\frac{2}{\sigma_Y \gamma}}}{1 - x^{\frac{2}{\sigma_Y \gamma}}} \right) + \left(\frac{\theta^{NSS}}{\psi} \right) \frac{1}{x(t)} \right] dt - x(t)\sigma_Y dB(t) \qquad \text{A.9}$$

From equations II. 12 and A. 8, we may observe that we have two mechanisms at play: one generating a negative correlation between international reserves and absorption (the SDE of equation II.12) and the other generating a positive correlation between absorption and reserves (the optimization equation A.8). This constitutes what is known as a feedback mechanism, which ensures that the economy is stable.

3. The Optimal Normalized Consumption and the Optimal Normalized Reserves:

3.1 the Martingale Approach:

In this analysis, we define the objective function as (see Appendix II.a):

$$V(x(0)) = \max_{c(s),x(\tau^{SS})} E\left\{ \int_0^{\tau^{SS}} e^{-\rho s} U(c(s))ds - U_x\left(x(\tau^{SS})\right) \right\} \qquad \text{A.11}$$

Here we maximize the discounted utility function of c(s), $e^{-\rho s}U(c(s))$, and minimize the utility function of the final value of the normalized reserves $U_x\left(x(\tau^{SS})\right)$. Through maximization, see Appendix II.a, we were able to determine the exact optimal values of c(t) and the corresponding optimal reserves $x(t)$. Unlike other methods, we do not give a prescribed desired value for $x(\tau^{SS})$. The derived equation for the optimal values is:

$$dc(t) = c(t)\left(\frac{1}{\gamma}\right)\left\{\left[r + \frac{(1+1/\gamma)}{2}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2 - \rho\right]dt + \frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}dB(t)\right\}, 0 < t \le \tau^{SS}$$

A.23

Equation A.23 shows that the optimal normalized consumption c(t) follows a Geometric Brownian motion with linear trend coefficient. For the estimated values of Egypt, the trend $\left(\frac{1}{\gamma}\right)\left[r + \frac{(1+1/\gamma)}{2}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2 - \rho\right] > 0$. The value of γ reflects the level of conservatism. As γ increases, we get conservative results.

The corresponding optimal values of normalized reserves x(t) are obtained by the substitution of eqn. A.23 into eqn. A. 3.

$$dx(t) = \left[x(t)(r - \mu_Y + \sigma_Y^2) + \left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(t)\right)\right] dt - x(t)\sigma_Y dB(t) \qquad \text{A.3}$$

Notice that $\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(t)\right)$>0, and this is a constraint on c(t). The value of $\left(\frac{\theta^{NSS}}{\psi}\right)$ is around "1" in most of the analysis. Remember that $\left(\frac{\theta^{NSS}}{\psi}\right) = A(t)/Y(t)$.

3.2 Portfolio Decision:

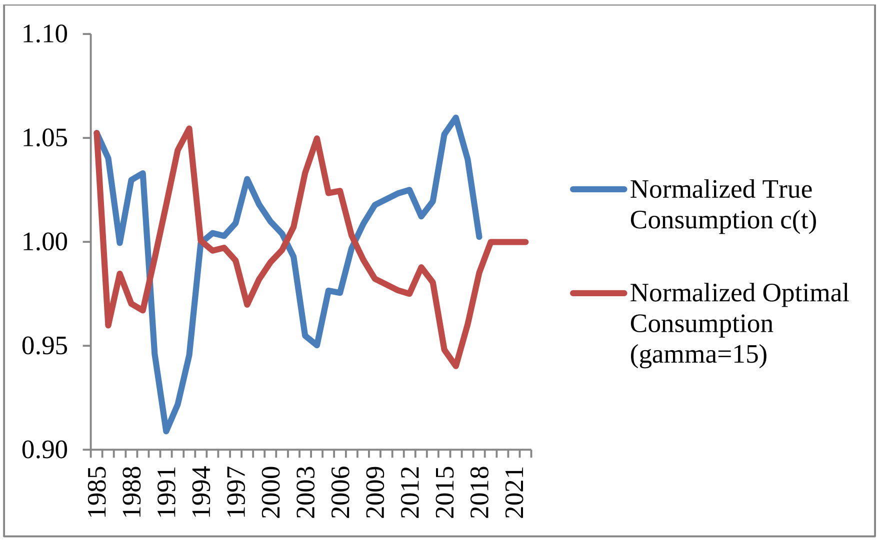
Egypt has had reserves in the year 2017 in the order of 14 percent of GDP. We need to gradually reduce this amount to a fixed level on the span of few years. At the same time, Egypt should buy risky assets with an amount that is enough to ensure the coverage of several months of imports once an SS occurs. The basic idea is that starting with high value of reserves X(t), we need to reduce this quantity gradually to a final fixed value X_C. At the time of SS, there is a payoff obtained from an external fund. The summation of the two should be close to a target value that might be equal, for example, to 10 months of imports. This policy will reduce the needed cashed reserves from the current high levels to the value X_C. The reduction in reserves will free needed cash to be infused in the economy. This will act as a stimulus to GDP growth.

4. RESULTS FOR THE EGYPTIAN ECONOMY:

In this section, we present the scenarios that result in optimal reserves, by the year 2021, to be 5 percent and 10 percent of GDP, and the corresponding optimal consumption. Other scenarios could have been obtained with the same approach. We compared the obtained optimal levels of consumption to the actual measured levels during the period 1985-2017. During this period, we used $\left(\frac{\theta^{NSS}}{\psi}\right) = A(t)/Y(t)$, with ψ =1. We also present the forecast till 2012. In the forecast, 2018-2012, we set $\left(\frac{\theta^{NSS}}{\psi}\right)$ of equation A.3 to be exactly 1. All the figures have the average values of c(t) and x(t).

We found that the optimal consumption was in the same order of magnitude as the actual one in most of the period under study, see Figure 2.

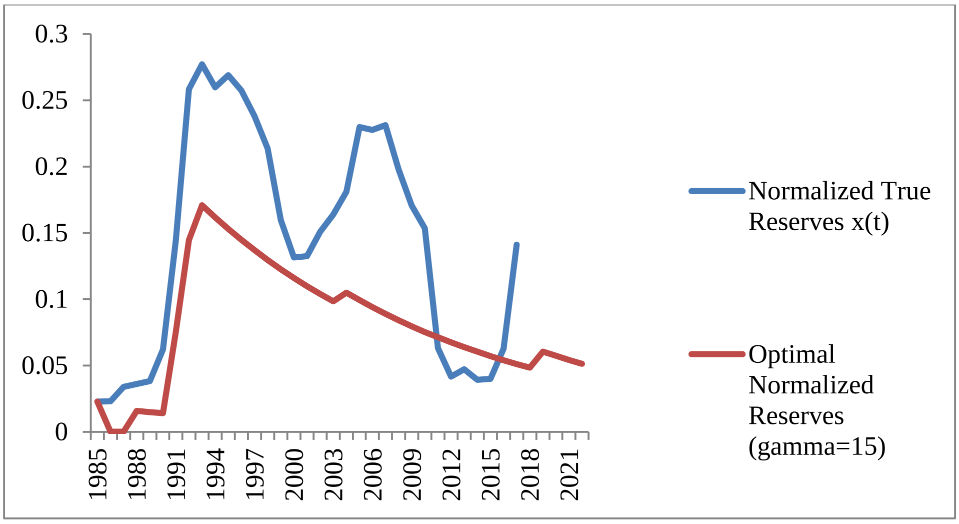
Figure 2. True and Optimal Normalized Consumption (gamma=15)



Notice that the optimal normalized consumption, c(t), is around 1. This is in agreement with the findings in (Caballero and Panageas 2004; Jeanne and Ranciere 2006).

We also found that the optimal levels of the normalized reserves are lower than the actual ones most of the time, see Figure 3.

Figure 3. True and Optimal Normalized Reserves (gamma=15)



The results in Figures. 2 and 3 were obtained by setting $\gamma = 15$.

For a more conservative approach, we present the same results but with $\gamma = 25$.

Figure 4. True and Optimal Normalized Consumption (gamma=25)

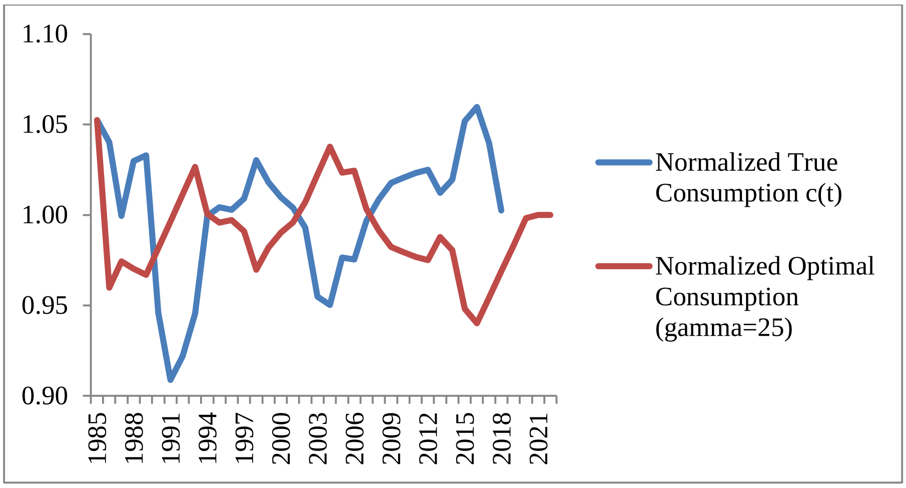


Figure 5. True and Optimal Normalized Reserves (gamma=25)

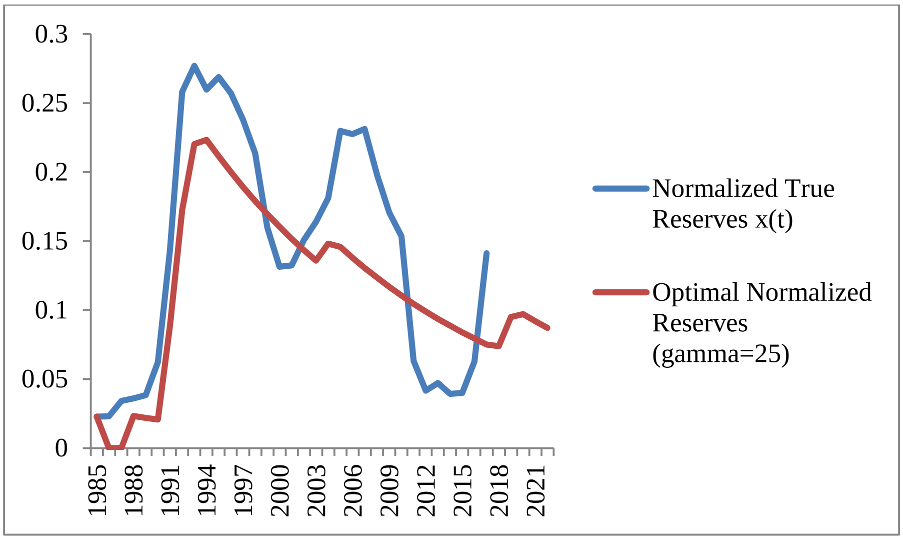


Figure 6. Comparison between Conservative (gamma=25) and risky (gamma=15) Optimal Consumption

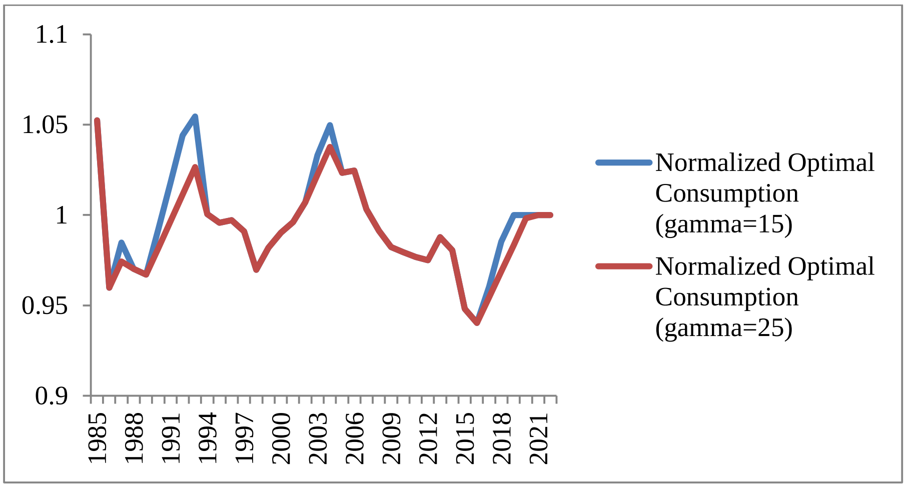
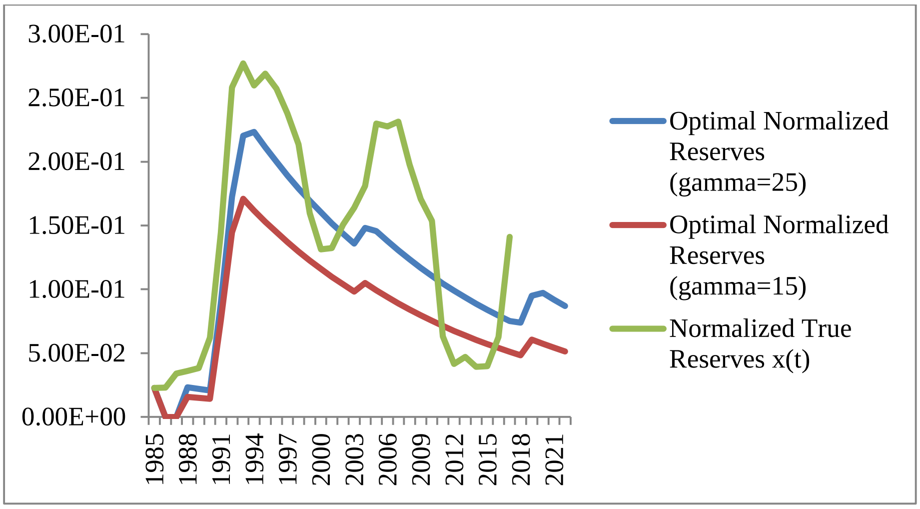


Figure 7. Comparison between Conservative (gamma=25) and risky (gamma=15) Optimal Reserves



Notice that in the conservative approach, $\gamma = 25$, and the risk taking approach, $\gamma = 15$, the levels of the normalized reserves are lower than the actual observed reserves (Figure 7).

Policly Implications:

We have developed a mathematical model that describes the behavior of reserves and consumption. The decision maker is able to change the different parameters and see their effects on the optimal values. The derived Optimal levels, with minimal assumptions, showed that, for Egypt, the policymaker decided to have reserves higher than the optimal values even for the risk aversion approach. On the other hand, by taking a more aggressive approach, the reserves would be freed and pumped into the economy to increase the GDP rate of growth.

APPENDICES TO PART II

Appendix II.a: Optimal Consumption

In this appendix, we derive the consumption C(t) that will optimize the expected value of the utility function subject to the system dynamics constraints. Instead of working with the reserves X(t), we shall, for mathematical purposes, use the normalized reserves, x(t), and the normalized consumption, c(t), defined as:

$$x(t) = \frac{X(t)}{\left(\psi Y(t)\right)} \qquad \text{(A. 1a)}$$

$$c(t) = \frac{C(t)}{\left(\psi Y(t)\right)} \qquad \text{(A. 1b)}$$

The GDP, Y(t), and the reserves, X(t), evolve according to the SDE's:

$$dY(t) = \mu_Y Y(t)dt + \sigma_Y Y(t)dB(t), \qquad 0 < r < \mu_Y \qquad \text{(II. 3)}$$

$$dX(t) = \left[rX(t) - C(t) + \theta^{NSS} Y(t)\right]dt \qquad \text{(II. 12)}$$

$$X(t) \geq 0$$

An SDE for x(t):

We shall derive an SDE for $x(t) = \frac{X(t)}{\left(\psi Y(t)\right)}$, for non-sudden stop, using Ito's lemma:

$$dx = \frac{\partial x}{\partial t}dt + \frac{\partial x}{\partial X}dX + \frac{\partial x}{\partial Y}dY + \frac{1}{2}\frac{\partial^2 x}{\partial X^2}\left(dX\right)^2 + \frac{\partial^2 x}{\partial X \partial Y}dXdY + \frac{1}{2}\frac{\partial^2 x}{\partial Y^2}\left(dY\right)^2 \qquad \text{(A. 2)}$$

Since $x(t) = \frac{X(t)}{\left(\psi Y(t)\right)}$

Then $\frac{\partial x}{\partial t} = 0$

$$\frac{\partial x}{\partial X} = \frac{1}{\psi Y}$$

$$\frac{\partial x}{\partial Y} = \frac{-\psi X}{\left(\psi Y\right)^2} = \frac{-X}{\psi Y^2}$$

$$\frac{\partial^2 x}{\partial X^2} = 0$$

$$\frac{\partial^2 x}{\partial X \partial Y} = \frac{-1}{\psi Y^2}$$

$$\frac{\partial^2 x}{\partial Y^2} = \frac{2X}{\psi Y^3}$$

$$(dY)^2 = \sigma_Y^2 Y^2 dt$$

$$(dX)^2 = 0$$

$$dXdY = 0$$

Substituting for the partial derivatives in equation A.2, we get:

$$dx = \frac{1}{\psi Y} dX - \frac{X}{\psi Y^2} dY - \frac{1}{\psi Y^2} dXdY + \frac{X}{\psi Y^3} (dY)^2$$

Substituting for the complete derivatives we get:

$$dx = \frac{1}{\psi Y}\left[rX(t) - C(t) + \theta^{NSS} Y(t)\right]dt - \frac{X}{\psi Y^2}\left[\mu_Y Y(t)dt + \sigma_Y Y(t)dB(t)\right] + 0 + \frac{X}{\psi Y^3}\sigma_Y^2 Y^2 dt$$

Which is reduced to:

$$dx = \frac{X}{\psi Y(t)}\left[r - \frac{C(t)}{X(t)} + \frac{\theta^{NSS} Y(t)}{X(t)}\right]dt - \frac{X(t)}{\psi Y(t)}\left[\mu_Y dt + \sigma_Y dB(t)\right] + \frac{X(t)}{\psi Y(t)}\sigma_Y^2 dt$$

Rearrange, we get:

$$dx = x(t)\left[r - \frac{C(t)}{X(t)} + \frac{\theta^{NSS} Y(t)}{X(t)}\right]dt - x(t)\left[\mu_Y dt + \sigma_Y dB(t)\right] + x(t)\sigma_Y^2 dt$$

$$= x(t)\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{C(t)}{X(t)} + \frac{\theta^{NSS} \psi Y(t)}{\psi X(t)}\right]dt - x(t)\sigma_Y dB(t)$$

$$= x(t)\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{c(t)}{x(t)} + \left(\frac{\theta^{NSS}}{\psi}\right)\frac{1}{x(t)}\right]dt - x(t)\sigma_Y dB(t)$$

Finally, $$dx(t) = \left[\left(\frac{\theta^{NSS}}{\psi}\right) + x(t)\left(r - \mu_Y + \sigma_Y^2\right) - c(t)\right]dt - x(t)\sigma_Y dB(t) \quad \text{(A. 3)}$$

Equation A.3 is the desired SDE that describes the evolution of the normalized reserves *x*(t).

Exact Solution for the Normalized variables:

Because all the components are explicitly independent of time and the diffusion is independent of the control, we will be able to derive an exact ODE for the control; c(x) (Abutaleb and Papaioannou 2019, Ch. 4; Mangel 1985, Ch. 2). The SDE describing the normalized reserves is given as:

$$dx(t) = x(t)\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{c(t)}{x(t)} + \frac{1}{x(t)}\right]dt - x(t)\sigma_Y dB(t) \qquad \text{(A. 3)}$$

Where we have set $\left(\frac{\theta^{NSS}}{\psi}\right) = 1$

The normalized utility function is given as:

$$U(c(x)) = \frac{c^{1-\gamma}}{1-\gamma},\ \gamma > 0, \gamma \neq 1 \qquad \text{(A. 4)}$$

The objective function is given as:

$$V(x(t)) = \max_{c(s)} E\left\{\int_t^{\tau^{SS}} U(c(s))ds + \varphi_1\left(x(\tau^{SS})\right)\right\} \qquad \text{(A. 5)}$$

Where all the variables involved are the normalized variables and where we have eliminated the discount factor $e^{-r(s-t)}$ to find a closed form solution and for simplifications.

The optimal normalized consumption c(t) satisfies the ordinary differential equation (ODE) (Abutaleb and Papaioannou 2019):

$$\frac{dc}{dx}\left[\frac{dF^u/dc}{db/dc}\frac{d^2b}{dc^2} - \frac{d^2F^u}{dc^2}\right] = \frac{2}{\sigma(x)}\left[\frac{b\,dF^u/dc}{db/dc} - F^u\right] + \frac{d^2F^u}{dcdx} - \frac{dF^u/dc}{db/dc}\frac{d^2b}{dcdx}$$

A.6

Where u(x)=c(x)

$$F^u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0, \gamma \neq 1$$

$$b(x,c) = x\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{c}{x} + \frac{1}{x}\right]$$

$$\sigma(x) = -x\sigma_Y$$

We need the following derivatives:

$$\frac{dF^u(c)}{dc} = c^{-\gamma}$$

$$\frac{d^2F^u(c)}{dc^2} = -\gamma\, c^{-\gamma-1}$$

$$\frac{d^2F^u(c)}{dcdx} = 0$$

$$\frac{db(x,c)}{dc} = -1$$

$$\frac{d^2b(x,c)}{dc^2} = 0$$

$$\frac{d^2b(x,c)}{dcdx} = 0$$

Substituting the different elements in equation A.6, we get:

$$\frac{dc}{dx}\left[\gamma\, c^{-\gamma-1}\right] = -\frac{2}{x\sigma_Y}\left[-x\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{c}{x} + \frac{1}{x}\right]c^{-\gamma} - \frac{c^{1-\gamma}}{1-\gamma}\right]$$

Dividing both sides by $c^{-\gamma}$ and rearrange, we get:

$$\frac{dc}{dx} = \frac{2c}{x\sigma_Y\gamma}\left[x\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{c}{x} + \frac{1}{x}\right] + \frac{c}{1-\gamma}\right]$$

$$= \frac{2c}{\sigma_Y\gamma}\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{c}{x} + \frac{1}{x}\right] + \frac{2c^2}{x\sigma_Y\gamma(1-\gamma)}$$

$$= \frac{2c}{\sigma_Y \gamma}\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{c}{x} + \frac{1}{x} + \frac{c}{x(1-\gamma)}\right]$$

$$= \frac{2c}{\sigma_Y \gamma}\left[\left(r - \mu_Y + \sigma_Y^2\right) + \frac{-c(1-\gamma) + (1-\gamma) + c}{x(1-\gamma)}\right]$$

$$= \frac{2c}{\sigma_Y \gamma}\left[\left(r - \mu_Y + \sigma_Y^2\right) + \frac{\gamma c + (1-\gamma)}{x(1-\gamma)}\right]$$

$$= \frac{2c}{\sigma_Y \gamma}\left[\left(r - \mu_Y + \sigma_Y^2\right) + \frac{\gamma c + (\gamma - 1)}{x(\gamma - 1)}\right] \qquad \text{(A. 7)}$$

$$= \frac{2c}{\sigma_Y \gamma}\left[\left(r - \mu_Y + \sigma_Y^2\right) + \frac{\gamma c}{x(\gamma - 1)} + \frac{(\gamma - 1)}{x(\gamma - 1)}\right]$$

$$\frac{dc}{dx} = \frac{2c}{\sigma_Y \gamma}\left[\left(r - \mu_Y + \sigma_Y^2\right) + \left(\frac{c}{x}\right)\frac{\gamma}{(\gamma - 1)} + \frac{1}{x}\right]$$

This is the desired ODE for the optimal normalized consumption c(x) as function of the normalized reserves.

Some approximations:

We know that *c* is in the order of 1, while *x* is in the order 0.2, and assuming that γ >>1 and $\left(r - \mu_Y + \sigma_Y^2\right) << \left(\frac{1}{x}\right)$, then

$$\frac{dc}{dx} \approx \frac{2c}{\sigma_Y \gamma}\left(\frac{c+1}{x}\right)$$

Separating variables, we get:

$$\frac{dc}{c(c+1)} \approx \frac{2}{\sigma_Y \gamma}\left(\frac{dx}{x}\right)$$

Which has the solution:

$$\int \frac{dc}{c(c+1)} = \ln c - \ln(c+1) + \ln K = \ln \frac{Kc}{c+1} \approx \frac{2}{\sigma_Y \gamma} \ln x$$

where K is the constant of integration, and is determined from the boundary conditions.

Thus $\frac{Kc}{c+1} \approx x^{\frac{2}{\sigma_Y \gamma}}$

which yields $c = \frac{x^{\frac{2}{\sigma_Y \gamma}}}{K - x^{\frac{2}{\sigma_Y \gamma}}}$,

A better approximate formula is $c = K_1 + K_2 \frac{x^{\frac{2}{\sigma_Y \gamma}}}{1 - x^{\frac{2}{\sigma_Y \gamma}}}$ (A. 8)

Substitute in the normalized reserves equation we get:

$$dx(t) = x(t)\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{1}{x(t)}\left(K_1 + K_2 \frac{x^{\frac{2}{\sigma_Y \gamma}}}{1 - x^{\frac{2}{\sigma_Y \gamma}}}\right) + \left(\frac{\theta^{NSS}}{\psi}\right)\frac{1}{x(t)}\right]dt - x(t)\sigma_Y dB(t)$$

(A.9)

In the simulation, we define the desired final value of $x(\tau^{ss})$ and find the corresponding optimal values for K_1 and K_2 that satisfy, in the minimum squared error sense, this final value. We also estimate the optimal normalized reserves through the SDE of $x(t)$, and the consumption that satisfies the equation $c(t) = K_1 + K_2 \frac{x^{\frac{2}{\sigma_Y \gamma}}}{1 - x^{\frac{2}{\sigma_Y \gamma}}}$.

The Martingale Approach with c(t):

In this subsection, we use the martingale optimality principle (Zhang 2007; Abutaleb 2013; Abutaleb and Papaioannou 2019), to find the optimal values of the normalized consumption and consequently the optimal levels of the normalized reserves.

The system dynamics is:

$$dx(t) = x(t)\left[\left(r - \mu_Y + \sigma_Y^2\right) - \frac{c(t)}{x(t)} + \left(\frac{\theta^{NSS}}{\psi}\right)\frac{1}{x(t)}\right]dt - x(t)\sigma_Y dB(t)$$

(A. 3)

$$= \left[x(t)\left(r - \mu_Y + \sigma_Y^2\right) + \left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(t)\right)\right]dt - x(t)\sigma_Y dB(t)$$

The normalized utility function is given as:

$$U(c(t)) = \frac{c^{1-\gamma}}{1-\gamma}, \ \gamma > 0, \gamma \neq 1 \tag{A. 10}$$

In this analysis, we define the objective function as:

$$V(x(0)) = \max_{c(s), x(\tau^{SS})} E\left\{ \int_0^{\tau^{SS}} e^{-\rho s} U(c(s))ds - U_x(x(\tau^{SS})) \right\} \tag{A. 11}$$

Notice that $U_x(x(\tau^{SS}))$ is not defined because it will be of no use for the derivation of the SDE of the optimal c(t).

In the martingale approach, we need to find the process H(t) such that $H(t)x(t) - H(t)\int \left(\left(\frac{\theta^{NSS}}{\psi} \right) - c(s) \right) ds$ is a martingale. Assume that H(t) has the SDE:

$$\frac{dH(t)}{H(t)} = \alpha_H dt + \beta_H dB(t) \tag{A. 12}$$

where B(t) is a Brownian motion.

Using Ito Lemma, we get:

$$d(Hx) = Hdx + xdH + dHdx$$

$$= H(t)\left\{ \left[x(t)(r - \mu_Y + \sigma_Y^2) + \left(\left(\frac{\theta^{NSS}}{\psi} \right) - c(t) \right) \right] dt - x(t)\sigma_Y dB(t) \right\} + x(t)H(t)[\alpha_H dt + \beta_H dB]$$

$$+ H(t)[\alpha_H dt + \beta_H dB]\left\{ \left[x(t)(r - \mu_Y + \sigma_Y^2) + \left(\left(\frac{\theta^{NSS}}{\psi} \right) - c(t) \right) \right] dt - x(t)\sigma_Y dB(t) \right\}$$

$$= H\left\{ \left[x(t)(r - \mu_Y + \sigma_Y^2) + \left(\left(\frac{\theta^{NSS}}{\psi} \right) - c(t) \right) \right] dt - x(t)\sigma_Y dB(t) \right\} + xH[\alpha_H dt + \beta_H dB] - Hx\beta_H \sigma_Y dt$$

Collecting terms, and dropping the dependence on t, we get:

$$d(Hx) = H\left[x(r - \mu_Y + \sigma_Y^2) + \left(\left(\frac{\theta^{NSS}}{\psi} \right) - c \right) + x\alpha_H - x\beta_H \sigma_Y \right] dt + xH(-\sigma_Y + \beta_H)dB$$

Moving $\left(\left(\frac{\theta^{NSS}}{\psi} \right) - c \right)$ dt to the left hand side we get:

$$d(Hx)-H\left(\left(\frac{\theta^{NSS}}{\psi}\right)-c\right)dt = H\left[x\left(r-\mu_Y+\sigma_Y^2\right)+x\alpha_H - x\beta_H\sigma_Y\right]dt + xH\left(-\sigma_Y+\beta_H\right)dB$$

For $Hx - H\int\left(\left(\frac{\theta^{NSS}}{\psi}\right)-c\right)ds$ a martingale, we need the drift term to be 0.

Thus, $$d(Hx)-H\left(\left(\frac{\theta^{NSS}}{\psi}\right)-c\right)dt = xH\left(-\sigma_Y+\beta_H\right)dB \qquad \text{A.13}$$

and $$0 = H\left[x\left(r-\mu_Y+\sigma_Y^2\right)+x\alpha_H - x\beta_H\sigma_Y\right]$$

This suggests that:

$$x\beta_H\sigma_Y = \left[x\left(r-\mu_Y+\sigma_Y^2\right)+x\alpha_H\right] = x\left[\left(r-\mu_Y+\sigma_Y^2\right)+\alpha_H\right]$$

Setting $\alpha_H = -r$, we get:

i.e. $$\beta_H = \frac{\left(-\mu_Y+\sigma_Y^2\right)}{\sigma_Y}$$

Thus, $\beta_H = \frac{\left(-\mu_Y+\sigma_Y^2\right)}{\sigma_Y}, \alpha_H = -r$, A.14

And $$d(Hx)-H\left(\left(\frac{\theta^{NSS}}{\psi}\right)-c\right)dt = xH\left(-\sigma_Y+\beta_H\right)dB$$

and the SDE for H(t) becomes:

$$\frac{dH(t)}{H(t)} = -rdt - \frac{\left(\mu_Y-\sigma_Y^2\right)}{\sigma_Y}dB \qquad \text{A.15}$$

which has the solution:

$$H(t) = e^{-\int_0^t rds}\exp\left\{\frac{-1}{2}\int_0^t\left[\frac{\left(\mu_Y-\sigma_Y^2\right)}{\sigma_Y}\right]^2 ds - \int_0^t \frac{\left(\mu_Y-\sigma_Y^2\right)}{\sigma_Y}dB(s)\right\} \qquad \text{A.16}$$

Notice that E{H(t)}=1 for all values of t.

The new Optimization Problem:

The new system dynamics becomes:

$$d(Hx) - H\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c\right)dt = xH(-\sigma_Y + \beta_H)dB$$

A.17

Integrating both sides, between 0 and τ^{SS}, we get:

$$H(\tau^{SS})x(\tau^{SS}) - H(0)x(0) - \int_0^{\tau^{SS}} H(s)\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(s)\right)ds = \int_0^{\tau^{SS}} x(s)H(s)(-\sigma_Y + \beta_H)dB(s)$$

i.e.

$$H(\tau^{SS})x(\tau^{SS}) - \int_0^{\tau^{SS}} H(s)\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(s)\right)ds = H(0)x(0) + \int_0^{\tau^{SS}} x(s)H(s)(-\sigma_Y + \beta_H)dB(s)$$

Taking the expectation of both sides, we get:

$$E\left\{H(\tau^{SS})x(\tau^{SS}) - \int_0^{\tau^{SS}} H(s)\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(s)\right)ds\right\} = E\{H(0)x(0)\}$$

$$= x(0)E\{H(0)\}$$

$$= x(0)$$

A.18

where we used the fact that E{H(0)}=1.

The optimization problem could now be stated as follows:

Find c(t) that maximizes $V(c(0))$:

$$V(x(0)) = \max_{c(s), x(\tau^{SS})} E\left\{\int_0^{\tau^{SS}} e^{-\rho s} U(c(s))ds - U_x(x(\tau^{SS}))\right\}$$

A.11

Subject to the constraint:

$$E\left\{H(\tau^{SS})x(\tau^{SS}) - \int_0^{\tau^{SS}} H(s)\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(s)\right)ds\right\} = E\{x(0)\}$$

A.18

With $$U(c(t)) = \frac{c^{1-\gamma}}{1-\gamma},\ \gamma > 0, \gamma \neq 1$$ A.10

Using the method of the Lagrange multiplier we need to find:

$$\max_{c(s),x(\tau^{SS})} E\left\{\int_0^{\tau^{SS}} e^{-\rho s}\frac{c(s)^{1-\gamma}}{1-\gamma}ds - U_x\left(x(\tau^{SS})\right) - \Lambda\left[H(\tau^{SS})x(\tau^{SS}) - \int_0^{\tau^{SS}} H(s)\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(s)\right)ds - x(0)\right]\right\}$$

Which has the form:

$$\max_{c(s),x(\tau^{SS})} E\left\{\int_0^{\tau^{SS}}\left[e^{-\rho s}\frac{c(s)^{1-\gamma}}{1-\gamma} + \Lambda H(s)\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(s)\right)\right]ds - U_x\left(x(\tau^{SS})\right) - \Lambda\left[H(\tau^{SS})x(\tau^{SS}) - x(0)\right]\right\} = 0$$

where Λ is the Lagrange multiplier. Assuming that the conditions for the exchange of derivative and expectation are satisfied, taking the derivative for c(t), we get:

$$\frac{\partial}{\partial c(s)} E\left\{\int_0^{\tau^{SS}}\left[e^{-\rho s}\frac{c(s)^{1-\gamma}}{1-\gamma} + \Lambda H(s)\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(s)\right)\right]ds\right\} = 0$$

i.e. $$\frac{\partial}{\partial c(s)}\left[e^{-\rho s}\frac{c(s)^{1-\gamma}}{1-\gamma} + \Lambda H(s)\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(s)\right)\right] = 0$$

which yields:

$$e^{-\rho s}c(s)^{-\gamma} - \Lambda H(s) = 0$$

i.e. $$c(s)^{-\gamma} = e^{\rho s}\Lambda H(s)$$

Taking the ln of both sides we get: $-\gamma \ln c(s) = \rho s + \ln \Lambda H(s)$

and $$\ln c(s) = -\frac{\rho s}{\gamma} + \left(\frac{-1}{\gamma}\right)\ln \Lambda H(s)$$

Thus, $c(t) = e^{-\rho t/\gamma}\left(\Lambda\right)^{(-1/\gamma)}\left(H(t)\right)^{(-1/\gamma)},\ 0 < t \le \tau^{SS}$ A.19

Notice that Λ is a constant deterministic value.

An SDE for c(t):

We now derive an SDE for the optimal normalized c(t), 0<t<= τ^{SS} . Since

$$c(t) = e^{-\rho t/\gamma}\left(\Lambda\right)^{(-1/\gamma)}\left(H(t)\right)^{(-1/\gamma)},\ 0 < t \le \tau^{SS} \quad \text{A.19}$$

then $dc(t) = e^{-\rho t/\gamma}\left(\Lambda\right)^{(-1/\gamma)}d\left(H(t)\right)^{(-1/\gamma)} + \left(-\rho/\gamma\right)e^{-\rho t/\gamma}\left(\Lambda\right)^{(-1/\gamma)}\left(H(t)\right)^{(-1/\gamma)}dt,\ 0 < t \le \tau^{SS}$

A.20

Since $\frac{dH(t)}{H(t)} = -rdt - \frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y} dB$ A.15

Define $y = H^{\alpha}$. Using Ito Lemma we get:

$$dy = \frac{\partial H^{\alpha}}{\partial H} dH + \frac{1}{2}\frac{\partial^2 H^{\alpha}}{\partial H^2}(dH)^2$$

$$= \alpha H^{\alpha-1} H\left(-rdt - \frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y} dB\right) + \frac{1}{2}\alpha(\alpha-1)H^{\alpha-2}H^2\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2 dt$$

$$= \alpha H^{\alpha}\left(-rdt - \frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y} dB\right) + \frac{1}{2}\alpha(\alpha-1)H^{\alpha}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2 dt$$

Finally we get:

$$d(H(t))^{\alpha} = -\alpha(H(t))^{\alpha}\left\{\left[r + \frac{(1-\alpha)}{2}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2\right]dt + \frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y} dB(t)\right\}$$ A.21

Setting $\alpha = (-1/\gamma)$, we get:

$$d(H(t))^{(-1/\gamma)} = (1/\gamma)(H(t))^{(-1/\gamma)}\left\{\left[r + \frac{(1+1/\gamma)}{2}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2\right]dt + \frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y} dB(t)\right\}$$

A.22

Substitute "$d(H(t))^{(-1/\gamma)}$"into the equation of $dc(t)$ A.20, we get:

$$dc(t) = e^{-\rho t/\gamma}(\Lambda)^{(-1/\gamma)}(1/\gamma)(H(t))^{(-1/\gamma)}\left\{\left[r + \frac{(1+1/\gamma)}{2}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2\right]dt + \frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y} dB(t)\right\}$$
$$+ (-\rho/\gamma)e^{-\rho t/\gamma}(\Lambda)^{(-1/\gamma)}(H(t))^{(-1/\gamma)}dt,\ 0 < t \le \tau^{SS}$$

Collecting terms, we get:

$$dc(t) = e^{-\rho t/\gamma}(\Lambda)^{(-1/\gamma)}(1/\gamma)(H(t))^{(-1/\gamma)}\left\{\left[r + \frac{(1+1/\gamma)}{2}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2 - \rho\right]dt + \frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y} dB(t)\right\}$$
$$0 < t \le \tau^{SS}$$

Since $c(t) = e^{-\rho t/\gamma}(\Lambda)^{(-1/\gamma)}(H(t))^{(-1/\gamma)},\ 0 < t \le \tau^{SS}$

Then $dc(t) = c(t)\left(\frac{1}{\gamma}\right)\left\{\left[r + \frac{(1+1/\gamma)}{2}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2 - \rho\right]dt + \frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}dB(t)\right\}, 0 < t \leq \tau^{SS}$

A.23

Equation A.23 shows that the optimal normalized consumption c(t), follows a Geometric Brownian motion with linear trend coefficient $\left(\frac{1}{\gamma}\right)\left[r + \frac{(1+1/\gamma)}{2}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2 - \rho\right]$. For the estimated values of Egypt, $\left(\frac{1}{\gamma}\right)\left[r + \frac{(1+1/\gamma)}{2}\left(\frac{(\mu_Y - \sigma_Y^2)}{\sigma_Y}\right)^2 - \rho\right] > 0$. The value of γ reflects the level of conservatism. As γ increase we get conservative results.

The corresponding optimal values of the normalized reserves x(t) are obtained by the substitution of eqn. A. 23 into eqn. A. 3.

$$dx(t) = \left[x(t)(r - \mu_Y + \sigma_Y^2) + \left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(t)\right)\right]dt - x(t)\sigma_Y dB(t) \qquad \text{A.3}$$

Notice that $\left(\left(\frac{\theta^{NSS}}{\psi}\right) - c(t)\right)$ >0 , and this is a constraint on c(t).

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